A superlinear lower bound on the number of 5-holes

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Abstract

Let $P$ be a finite set of points in the plane in general position, that is, no three points of $P$ are on a common line. We say that a set $H$ of five points from $P$ is a 5-hole in $P$ if $H$ is the vertex set of a convex 5-gon containing no other points of $P$. For a positive integer $n$, let $h_5(n)$ be the minimum number of 5-holes among all sets of $n$ points in the plane in general position.

Despite many efforts in the last 30 years, the best known asymptotic lower and upper bounds for $h_5(n)$ have been of order $\Omega(n)$ and $O(n^2)$, respectively. We show that $h_5(n) = \Omega(n \log^{4/5} n)$, obtaining the first superlinear lower bound on $h_5(n)$.

The following structural result, which might be of independent interest, is a crucial step in the proof of this lower bound. If a finite set $P$ of points in the plane in general position is partitioned by a line $\ell$ into two subsets, each of size at least 5 and not in convex position, then $\ell$ intersects the convex hull of some 5-hole in $P$. The proof of this result is computer-assisted.

1 Introduction

We say that a set of points in the plane is in general position if it contains no three points on a common line. A point set is in convex position if it is the vertex set of a convex polygon. In 1935, Erdős and Szekeres [15] proved the following theorem, which is a classical result both in combinatorial geometry and Ramsey theory.

Theorem ([15], The Erdős–Szekeres Theorem). For every integer $k \geq 3$, there is a smallest integer $n = n(k)$ such that every set of at least $n$ points in general position in the plane contains $k$ points in convex position.

The Erdős–Szekeres Theorem motivated a lot of further research, including numerous modifications and extensions of the theorem. Here we mention only results closely related to the main topic of our paper.

Let $P$ be a finite set of points in general position in the plane. We say that a set $H$ of $k$ points from $P$ is a $k$-hole in $P$ if $H$ is the vertex set of a convex $k$-gon containing no other

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points of \( P \). In the 1970s, Erdős [13] asked whether, for every positive integer \( k \), there is a \( k \)-hole in every sufficiently large finite point set in general position in the plane. Harborth [20] proved that there is a 5-hole in every set of 10 points in general position in the plane and gave a construction of 9 points in general position with no 5-hole. After unsuccessful attempts of researchers to answer Erdős’ question affirmatively for any fixed integer \( k \geq 6 \), Horton [21] constructed, for every positive integer \( n \), a set of \( n \) points in general position in the plane with no 7-hole. His construction was later generalized to so-called Horton sets and squared Horton sets [29] and to higher dimensions [30]. The question whether there is a 6-hole in every sufficiently large finite planar point set remained open until 2007 when Gerken [18] and Nicolás [22] independently gave an affirmative answer.

For positive integers \( n \) and \( k \), let \( h_k(n) \) be the minimum number of \( k \)-holes in a set of \( n \) points in general position in the plane. Due to Horton’s construction, \( h_k(n) = 0 \) for every \( n \) and every \( k \geq 7 \). Asymptotically tight estimates for the functions \( h_3(n) \) and \( h_4(n) \) are known. The best known lower bounds are due to Aichholzer et al. [4] who showed that \( h_3(n) \geq n^2 - \frac{32}{7}n + \frac{22}{7} \) and \( h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - o(n) \). The best known upper bounds \( h_3(n) \leq 1.6196n^2 + o(n^2) \) and \( h_4(n) \leq 1.9397n^2 + o(n^2) \) are due to Bárány and Valtr [11].

For \( h_5(n) \) and \( h_6(n) \), no matching bounds are known. So far, the best known asymptotic upper bounds on \( h_5(n) \) and \( h_6(n) \) were obtained by Bárány and Valtr [11] and give \( h_5(n) \leq 1.0207n^2 + o(n^2) \) and \( h_6(n) \leq 0.2006n^2 + o(n^2) \). For the lower bound on \( h_6(n) \), Valtr [31] showed \( h_6(n) \geq n/229 - 4 \).

In this paper we give a new lower bound on \( h_5(n) \). It is widely conjectured that \( h_5(n) \) grows quadratically in \( n \), but to this date only lower bounds on \( h_5(n) \) that are linear in \( n \) have been known. As noted by Bárány and Füredi [9], a linear lower bound of \( \lfloor n/10 \rfloor \) follows directly from Harborth’s result [20]. Bárány and Károlyi [10] improved this bound to \( h_5(n) \geq n/6 - O(1) \). In 1987, Dehnhardt [13] showed \( h_5(11) = 2 \) and \( h_5(12) = 3 \), obtaining \( h_5(n) \geq 3 \lfloor n/12 \rfloor \). However, his result remained unknown to the scientific community until recently. Garcia [17] then presented a proof of the lower bound \( h_5(n) \geq 3 \lfloor \frac{n - 1}{3} \rfloor \) and a slightly better estimate \( h_5(n) \geq \lceil 3/7(n - 11) \rceil \) was shown by Aichholzer, Hackl, and Vogtenhuber [5]. Quite recently, Valtr [31] obtained \( h_5(n) \geq n/2 - O(1) \). This was strengthened by Aichholzer et al. [4] to \( h_5(n) \geq 3n/4 - o(n) \). All improvements on the multiplicative constant were achieved by utilizing the values of \( h_5(10), h_5(11), \) and \( h_5(12) \). In the bachelor’s thesis of Scheucher [26] the exact values \( h_5(13) = 3, h_5(14) = 6, \) and \( h_5(15) = 9 \) were determined and \( h_5(16) \in \{10, 11\} \) was shown. During the preparation of this paper, we further determined the value \( h_5(16) = 11 \); see our webpage [25]. Table 1 summarizes our knowledge on the values of \( h_5(n) \) for \( n \leq 20 \). The values \( h_5(n) \) for \( n \leq 16 \) can be used to obtain further improvements on the multiplicative constant. By revising the proofs of [1] Lemma 1] and [4] Theorem 3], one can obtain \( h_5(n) \geq n - 10 \) and \( h_5(n) \geq 3n/2 - o(n) \), respectively. We also note that it was shown in [24] that if \( h_3(n) \geq (1 + \epsilon)n^2 - o(n^2) \), then \( h_5(n) = \Omega(n^2) \).

\[
\begin{array}{c|cccccccccccccccc}
 n & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
 h_5(n) & 0 & 1 & 2 & 3 & 3 & 6 & 9 & 11 & \leq 16 & \leq 21 & \leq 26 & \leq 33 \\
\end{array}
\]

**Table 1:** The minimum number \( h_5(n) \) of 5-holes determined by any set of \( n \leq 20 \) points.

As our main result, we give the first superlinear lower bound on \( h_5(n) \). This solves an open problem, which was explicitly stated, for example, in a book by Brass, Moser, and Pach [12] Chapter 8.4, Problem 5] and in the survey [2].
Theorem 1. There is an absolute constant \( c > 0 \) such that for every integer \( n \geq 10 \) we have 
\[ h_5(n) \geq cn \log^{4/5} n. \]

Let \( P \) be a finite set of points in the plane in general position and let \( \ell \) be a line that contains no point of \( P \). We say that \( P \) is \( \ell \)-divided if there is at least one point of \( P \) in each of the two halfplanes determined by \( \ell \). For an \( \ell \)-divided set \( P \), we use \( P = A \cup B \) to denote the fact that \( \ell \) partitions \( P \) into the subsets \( A \) and \( B \). In the rest of the paper, we assume without loss of generality that \( \ell \) is vertical and directed upwards, \( A \) is to the left of \( \ell \), and \( B \) is to the right of \( \ell \).

The following result, which might be of independent interest, is a crucial step in the proof of Theorem 1.

Theorem 2. Let \( P = A \cup B \) be an \( \ell \)-divided set with \( |A|, |B| \geq 5 \) and with neither \( A \) nor \( B \) in convex position. Then there is an \( \ell \)-divided 5-hole in \( P \).

The proof of Theorem 2 is computer-assisted. We reduce the result to several statements about point sets of size at most 11 and then verify each of these statements by an exhaustive computer search. To verify the computer-aided proofs we have implemented two independent programs, which, in addition, are based on different abstractions of point sets; see Subsection 5.2. Some of the tools that we use originate from the bachelor’s theses of Scheucher [26, 27].

Using a result of García [17], we adapt the proof of Theorem 1 to provide improved lower bounds on the minimum numbers of 3-holes and 4-holes.

Theorem 3. The following two bounds are satisfied for every positive integer \( n \):

(i) \( h_3(n) \geq n^2 + \Omega(n \log^{2/3} n) \) and

(ii) \( h_4(n) \geq \frac{n^2}{2} + \Omega(n \log^{3/4} n) \).

In the rest of the paper, we assume that every point set \( P \) is planar, finite, and in general position. We also assume, without loss of generality, that all points in \( P \) have distinct \( x \)-coordinates. We use \( \text{conv}(P) \) to denote the convex hull of \( P \) and \( \partial \text{conv}(P) \) to denote the boundary of the convex hull of \( P \).

A subset \( Q \) of \( P \) that satisfies \( P \cap \text{conv}(Q) = Q \) is called an island of \( P \). Note that every \( k \)-hole in an island \( Q \) of \( P \) is also a \( k \)-hole in \( P \). For any subset \( R \) of the plane, if \( R \) contains no point of \( P \), then we say that \( R \) is empty of points of \( P \).

In Section 2 we derive quite easily Theorem 1 from Theorem 2. Theorem 3 is proved in Section 3. Then, in Section 4, we give some preliminaries for the proof of Theorem 2, which is presented in Section 5. Finally, in Section 6 we give some final remarks. In particular, we show that the assumptions in Theorem 2 are necessary. To provide a better general view, we present a flow summary of the proof of Theorem 1 in Appendix A.

2 Proof of Theorem 1

We now apply Theorem 2 to obtain a superlinear lower bound on the number of 5-holes in a given set of \( n \) points. Without loss of generality, we assume that \( n = 2^t \) for some integer \( t \geq 5^5 \).
We prove by induction on \( t \geq 5^5 \) that the number of 5-holes in an arbitrary set \( P \) of \( n = 2^t \) points is at least \( f(t) := c \cdot 2^t 4^{t/5} \) for some absolute constant \( c > 0 \). For \( t = 5^5 \), we have \( n > 10 \) and, by the result of Harborth [20], there is at least one 5-hole in \( P \). If \( c \) is sufficiently small, then \( f(t) = c \cdot n \log_2^{4/5} n \leq 1 \) and we have at least \( f(t) \) 5-holes in \( P \), which constitutes our base case.

For the inductive step we assume that \( t > 5^5 \). We first partition \( P \) with a line \( \ell \) into two sets \( A \) and \( B \) of size \( n/2 \) each. Then we further partition \( A \) and \( B \) into smaller sets using the following well-known lemma, which is, for example, implied by a result of Steiger and Zhao [28, Theorem 1].

**Lemma 4** (28). Let \( P' = A' \cup B' \) be an \( \ell \)-divided set and let \( r \) be a positive integer such that \( r \leq |A'|, |B'| \). Then there is a line that is disjoint from \( P' \) and that determines an open halfplane \( h \) with \( |A' \cap h| = r = |B' \cap h| \).

We set \( r := \lfloor \log_2^{1/5} n \rfloor \), \( s := \lfloor n/(2r) \rfloor \), and apply Lemma 4 iteratively in the following way to partition \( P \) into islands \( P_1, \ldots, P_{s+1} \) of \( P \) so that the sizes of \( P_i \cap A \) and \( P_i \cap B \) are exactly \( r \) for every \( i \in \{1, \ldots, s\} \). Let \( P'_0 := P \). For every \( i = 1, \ldots, s \), we consider a line that is disjoint from \( P'_{i-1} \) and that determines an open halfplane \( h \) with \( |P'_{i-1} \cap A \cap h| = r = |P'_{i-1} \cap B \cap h| \). Such a line exists by Lemma 4 applied to the \( \ell \)-divided set \( P'_{i-1} \). We then set \( P_i := P'_{i-1} \setminus P_i \), and continue with \( i + 1 \). Finally, we set \( P_{s+1} := P'_s \).

For every \( i \in \{1, \ldots, s\} \), if one of the sets \( P_i \cap A \) and \( P_i \cap B \) is in convex position, then there are at least \( \binom{s}{i} 5 \)-holes in \( P_i \) and, since \( P_i \) is an island of \( P \), we have at least \( \binom{s}{i} 5 \)-holes in \( P \). If this is the case for at least \( s/2 \) islands \( P_i \), then, given that \( s = \lfloor n/(2r) \rfloor \) and thus \( s/2 \geq n/(4r) \), we obtain at least \( n/(4r) \binom{s}{i} \geq c \cdot n \log_2^{4/5} n \) 5-holes in \( P \) for a sufficiently small \( c > 0 \).

We thus further assume that for more than \( s/2 \) islands \( P_i \), neither of the sets \( P_i \cap A \) nor \( P_i \cap B \) is in convex position. Since \( r = \lfloor \log_2^{1/5} n \rfloor \geq 5 \), Theorem 2 implies that there is an \( \ell \)-divided 5-hole in each such \( P_i \). Thus there is an \( \ell \)-divided 5-hole in \( P_i \) for more than \( s/2 \) islands \( P_i \). Since each \( P_i \) is an island of \( P \) and since \( s = \lfloor n/(2r) \rfloor \), we have more than \( s/2 \geq n/(4r) \) \( \ell \)-divided 5-holes in \( P \). As \( |A| = |B| = n/2 = 2^{t-1} \), there are at least \( f(t-1) \) 5-holes in \( A \) and at least \( f(t-1) \) 5-holes in \( B \) by the inductive assumption. Since \( A \) and \( B \) are separated by the line \( \ell \), we have at least

\[
2f(t-1) + n/(4r) = 2c(n/2) \log_2^{4/5} (n/2) + n/(4r) \geq c(n/(4r)) \log_2^{4/5} (n/2) + n/(4r) \geq c n(t-1)^{4/5} + n/(4t^{1/5})
\]

5-holes in \( P \). The right side of the above expression is at least \( f(t) = c n^{t-1} \), because the inequality \( c n(t-1)^{4/5} + n/(4t^{1/5}) \geq c n^{t-1} \) is equivalent to the inequality \( (t-1)^{4/5}t^{4/5} + 1/(4c) \geq t \), which is true if \( c \) is sufficiently small, as \( (t-1)^{4/5}t^{4/5} \geq t-1 \). This finishes the proof of Theorem 1.

### 3 Proof of Theorem 3

In this section we improve the lower bounds on the minimum number of 3-holes and 4-holes. To this end we use the notion of generated holes as introduced by Garcia [17].

Given a 5-hole \( H \) in a point set \( P \), a 3-hole in \( P \) is *generated by \( H \) if it is spanned by the leftmost point \( p \) of \( H \) and the two vertices of \( H \) that are not adjacent to \( p \) on the boundary of \( \text{conv}(H) \). Similarly, a 4-hole in \( P \) is *generated by \( H \) if it is spanned by the vertices of \( H \).
with the exception of one of the points adjacent to the leftmost point of $H$ on the boundary of $\text{conv}(H)$. We call a 3-hole or a 4-hole in $P$ generated if it is generated by some 5-hole in $P$. We denote the number of generated 3-holes and generated 4-holes in $P$ by $h_{35}(P)$ and $h_{45}(P)$, respectively. We also denote by $h_{35}(n)$ and $h_{45}(n)$ the minimum of $h_{35}(P)$ and $h_{45}(P)$, respectively, among all sets $P$ of $n$ points.

For an integer $k \geq 3$ and a point set $P$, let $h_k(P)$ be the number of $k$-holes in $P$. García [17] proved the following relationships between $h_3(P)$ and $h_{35}(P)$ and between $h_4(P)$ and $h_{45}(P)$.

**Theorem 5 ([17]).** Let $P$ be a set of $n$ points and let $\gamma(P)$ be the number of extremal points of $P$. Then the following two equalities are satisfied:

(i) $h_3(P) = n^2 - 5n + \gamma(P) + 4 + h_{35}(P)$ and

(ii) $h_4(P) = \frac{n^2}{2} - \frac{7n}{2} + \gamma(P) + 3 + h_{45}(P)$.

The proofs of both parts of Theorem 3 are carried out by induction on $n$ similarly to the proof of Theorem 1. The base cases follow from the fact that each set $P$ of $n \geq 10$ points contains at least one 5-hole in $P$ and thus a generated 3-hole in $P$ and a generated 4-hole in $P$. For the inductive step, let $P = A \cup B$ be an $\ell$-divided set of $n$ points with $|A|, |B| \geq \left\lceil \frac{n}{4} \right\rceil$, where $n$ is a sufficiently large positive integer.

To show part (i), it suffices to prove $h_{35}(P) \geq \Omega(n \log^{2/3} n)$ as the statement then follows from Theorem 5. We use the recursive approach from the proof of Theorem 1 where we choose $r = \left\lfloor \log^{2/3} n \right\rfloor$. In each step of the recursion we either obtain $\left\lceil \frac{n}{4r} \right\rceil$ pairwise disjoint $r$-holes in $P$ or $\left\lfloor \frac{n}{4r} \right\rfloor$ pairwise disjoint $\ell$-divided 5-holes in $P$.

In the first case, each $r$-hole in $P$ admits $(\ell^3)$ 3-holes in $P$ and, by Theorem 5, it contains $(\ell^3) - r^2 + 5r - r - 4$ generated 3-holes in $P$. Thus, in total, we count at least $\frac{n}{4r} (\ell^3) - O(nr) \geq \Omega(n \log^{2/3} n)$ generated 3-holes in $P$.

In the second case, we have at least $\left\lceil \frac{n}{4r} \right\rceil$ $\ell$-divided 5-holes in $P$. Without loss of generality, we can assume that at least $\frac{1}{2} \left\lceil \frac{n}{4r} \right\rceil \geq \left\lceil \frac{n}{8r} \right\rceil$ of those $\ell$-divided 5-holes in $P$ contain at least two points to the right of $\ell$, as we otherwise continue with the horizontal reflection of $P$, which has $\ell$ as the axis of reflection. Therefore we have at least $\left\lceil \frac{n}{8r} \right\rceil$ $\ell$-divided generated 3-holes in $P$ and, analogously as in the proof of Theorem 1 we obtain

$$h_{35}(P) \geq 2h_{35} \left( \left\lceil \frac{n}{2} \right\rceil \right) + \left\lceil \frac{n}{4r} \right\rceil \geq \Omega(n \log^{2/3} n).$$

This finishes the proof of part (i).

The proof of part (ii) is almost identical. We choose $r = \left\lfloor \log^{1/4} n \right\rfloor$ and use the facts that every $r$-hole in $P$ contains $(\ell^4) - \frac{r^2}{2} + \frac{7r}{2} - r - 3$ generated 4-holes in $P$ and that every $\ell$-divided 5-hole in $P$ generates two 4-holes in $P$, at least one of which is $\ell$-divided. This finishes the proof of Theorem 3.

### 4 Preliminaries

Before proceeding with the proof of Theorem 2, we first introduce some notation and definitions, and state some immediate observations.

Let $a, b, c$ be three distinct points in the plane. We denote the line segment spanned by $a$ and $b$ as $ab$, the ray starting at $a$ and going through $b$ as $\overrightarrow{ab}$, and the line through $a$ and
b directed from a to b as \( \overline{ab} \). We say c is to the left (right) of \( \overline{ab} \) if the triple \( (a,b,c) \) traced in this order is oriented counterclockwise (clockwise). Note that c is to the left of \( \overline{ab} \) if and only if c is to the right of \( \overline{ba} \), and that the triples \( (a,b,c) \), \( (b,c,a) \), and \( (c,a,b) \) have the same orientation. We say a point set \( S \) is to the left (right) of \( \overline{ab} \) if every point of \( S \) is to the left (right) of \( \overline{ab} \).

**Sectors of polygons** For an integer \( k \geq 3 \), let \( P \) be a convex polygon with vertices \( p_1, p_2, \ldots, p_k \) traced counterclockwise in this order. We denote by \( S(p_1, p_2, \ldots, p_k) \) the open convex region to the left of each of the three lines \( p_1p_2 \), \( p_2p_k \), and \( p_{k-1}p_k \). We call the region \( S(p_1, p_2, \ldots, p_k) \) a sector of \( P \). Note that every convex \( k \)-gon defines exactly \( k \) sectors. Figure 1 gives an illustration.

![Figure 1](image)

**Figure 1:** (a) An example of sectors. (b) An example of \( a^* \)-wedges with \( t = |A| - 1 \). (c) An example of \( a^* \)-wedges with \( t < |A| - 1 \).

We use \( \triangle(p_1, p_2, p_3) \) to denote the closed triangle with vertices \( p_1, p_2, p_3 \). We also use \( \Box(p_1, p_2, p_3, p_4) \) to denote the closed quadrilateral with vertices \( p_1, p_2, p_3, p_4 \) traced in the counterclockwise order along the boundary.

The following simple observation summarizes some properties of sectors of polygons.

**Observation 6.** Let \( P = A \cup B \) be an \( \ell \)-divided set with no \( \ell \)-divided 5-hole in \( P \). Then the following conditions are satisfied.

(i) Every sector of an \( \ell \)-divided 4-hole in \( P \) is empty of points of \( P \).

(ii) If \( S \) is a sector of a 4-hole in \( A \) and \( S \) is empty of points of \( A \), then \( S \) is empty of points of \( B \).

**\( \ell \)-critical sets and islands** An \( \ell \)-divided set \( C = A \cup B \) is called \( \ell \)-critical if it fulfills the following two conditions.

(i) Neither \( A \) nor \( B \) is in convex position.

(ii) For every extremal point \( x \) of \( C \), one of the sets \( (C \setminus \{x\}) \cap A \) and \( (C \setminus \{x\}) \cap B \) is in convex position.

Note that every \( \ell \)-critical set \( C = A \cup B \) contains at least four points in each of \( A \) and \( B \). Figure 2 shows some examples of \( \ell \)-critical sets. If \( P = A \cup B \) is an \( \ell \)-divided set with neither \( A \) nor \( B \) in convex position, then there exists an \( \ell \)-critical island of \( P \). This can be seen by iteratively removing extremal points so that none of the parts is in convex position after the removal.
Observation 7. Let \( P = A \cup B \) be an \( \ell \)-divided set with \( A \) not in convex position. Then the points \( a_1, \ldots, a_{|A|-1} \) lie to the right of \( a^* \) and the points \( a_1, \ldots, a_{|A|-1} \) lie to the left of \( a^* \).

5 Proof of Theorem 2

First, we give a high-level overview of the main ideas of the proof of Theorem 2. We proceed by contradiction and we suppose that there is no \( \ell \)-divided 5-hole in a given \( \ell \)-divided set \( P = A \cup B \) with \( |A|, |B| \geq 5 \) and with neither \( A \) nor \( B \) in convex position. If \( |A|, |B| = 5 \), then the statement follows from the result of Harborth [20]. Thus we assume that \( |A| \geq 6 \) or \( |B| \geq 6 \). We reduce \( P \) to an island \( Q \) of \( P \) by iteratively removing points from the convex hull until one of the two parts \( Q \cap A \) and \( Q \cap B \) contains exactly five points or \( Q \) is \( \ell \)-critical with \( |Q \cap A|, |Q \cap B| \geq 6 \). If \( |Q \cap A| = 5 \) and \( |Q \cap B| \geq 6 \) or vice versa, then we reduce \( Q \) to an island of \( Q \) with eleven points and, using a computer-aided result (Lemma [14]), we show that there is an \( \ell \)-divided 5-hole in that island and hence in \( P \). If \( Q \) is \( \ell \)-critical with \( |Q \cap A|, |Q \cap B| \geq 6 \), then we show that \( |A \cap \partial \text{conv}(Q)|, |B \cap \partial \text{conv}(Q)| \leq 2 \) and that, if \( |A \cap \partial \text{conv}(Q)| = 2 \), then \( a^* \) is the single interior point of \( Q \cap A \) and similarly for \( B \) (Lemma [19]). Without loss of generality, we assume that \( |A \cap \partial \text{conv}(Q)| = 2 \) and thus \( a^* \) is the single interior point of \( Q \cap A \). Using this assumption, we prove that \( |Q \cap B| < |Q \cap A| \) (Proposition [21]).
exchanging the roles of $Q \cap A$ and $Q \cap B$, we obtain $|Q \cap A| \leq |Q \cap B|$ (Proposition 22), which gives a contradiction.

To bound $|Q \cap B|$, we use three results about the sizes of the parameters $w_1, \ldots, w_t$ for the $\ell$-divided set $Q$, that is, about the numbers of points of $Q \cap B$ in the $a^*$-wedges $W_1, \ldots, W_t$ of $Q$. We show that if we have $w_i = 2 = w_j$ for some $1 \leq i < j \leq t$, then $w_k = 0$ for some $k$ with $i < k < j$ (Lemma 12). Further, for any three or four consecutive $a^*$-wedges whose union is convex and contains at least four points of $Q \cap B$, each of those $a^*$-wedges contains at most two such points (Lemma 18). Finally, we show that $w_1, \ldots, w_t \leq 3$ (Lemma 20). The proofs of Lemmas 18 and 20 rely on some results about small $\ell$-divided sets with computer-aided proofs (Lemmas 15, 16, and 17). Altogether, this is sufficient to show that $|Q \cap B| < |Q \cap A|$. We now start the proof of Theorem 2 by showing that if there is an $\ell$-divided 5-hole in the intersection of $P$ with a union of consecutive $a^*$-wedges, then there is an $\ell$-divided 5-hole in $P$.

**Lemma 8.** Let $P = A \cup B$ be an $\ell$-divided set with $A$ not in convex position. For integers $i,j$ with $1 \leq i \leq j \leq t$, let $W := \bigcup_{k=i}^{j} W_k$ and $Q := P \cap W$. If there is an $\ell$-divided 5-hole in $Q$, then there is an $\ell$-divided 5-hole in $P$.

**Proof.** If $W$ is convex then $Q$ is an island of $P$ and the statement immediately follows. Hence we assume that $W$ is not convex. The region $W$ is bounded by the rays $a^*a_{i-1}$ and $a^*a_j$ and all points of $P \setminus Q$ lie in the convex region $\mathbb{R}^2 \setminus W$; see Figure 3.

![Figure 3: Illustration of the proof of Lemma 8.](image)

The point $a_j$ is to the right of $a^*$. The point $a_j$ is to the left of $a^*$.

(c) The hole $H$ properly intersects the ray $a^*a_j$. The boundary of the convex hull of $H$ is drawn red and the convex hull of $H'$ is drawn blue.

Since $W$ is non-convex and every $a^*$-wedge contained in $W$ intersects $\ell$, at least one of the points $a_{i-1}$ and $a_j$ lies to the left of $a^*$. Moreover, the points $a_i, \ldots, a_{j-1}$ are to the right of $a^*$ by Observation 7. Without loss of generality, we assume that $a_{i-1}$ is to the left of $a^*$.

Let $H$ be an $\ell$-divided 5-hole in $Q$. If $a_j$ is to the left of $a^*$, then we let $h$ be the closed halfplane determined by the vertical line through $a^*$ such that $a_{i-1}$ and $a_j$ lie in $h$. Otherwise, if $a_j$ is to the right of $a^*$, then we let $h$ be the closed halfplane determined by the line $a^*a_j$ such that $a_{i-1}$ lies in $h$. In either case, $h \cap A \cap Q = \{a^*, a_{i-1}, a_j\}$.

We say that $H$ properly intersects a ray $r$ if there are points $p, q \in H$ such that the interior of the segment $pq$ intersects $r$. Now we show that if $H$ properly intersects the ray $a^*a_j$, then $H$ contains $a_{i-1}$. Assume there are points $p, q \in H$ such that $pq$ properly intersects $r := a^*a_j$. Since $r$ lies in $h$ and neither of $p$ and $q$ lies in $r$, at least one of the points $p$ and $q$ lies in $h \setminus r$. 

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Without loss of generality, we assume \( p \in h \setminus r \). From \( h \cap A \cap Q = \{ a^*, a_{i-1}, a_j \} \) we have \( p = a_{i-1} \). By symmetry, if \( H \) properly intersects the ray \( a^*a_{i-1} \), then \( H \) contains \( a_j \).

Suppose for contradiction that \( H \) properly intersects both rays \( a^*a_{i-1} \) and \( a^*a_j \). Then \( H \) contains the points \( a_{i-1}, a_j, x, y, z \) for some points \( x, y, z \in Q \), where \( a_{i-1}x \) intersects \( a^*a_j \), and \( a_jz \) intersects \( a^*a_{i-1} \). Observe that \( z \) is to the left of \( a_{i-1}a^* \) and that \( x \) is to the right of \( a_ja^* \). If \( a_j \) lies to the right of \( a^* \), then \( z \) is to the left of \( a^* \), and thus \( z \) is in \( A \); see Figure 3(a). However, this is impossible as \( z \) also lies in \( h \). Hence, \( a_j \) lies to the left of \( a^* \); see Figure 3(b). As \( x \) and \( z \) are both to the right of \( a^* \), the point \( a^* \) is inside the convex quadrilateral \( \triangle(a_{i-1}, a_j, x, z) \). This contradicts the assumption that \( H \) is a 5-hole in \( Q \).

So assume that \( H \) properly intersects exactly one of the rays \( a^*a_{i-1} \) and \( a^*a_j \), say \( a^*a_j \); see Figure 3(c). In this case, \( H \) contains \( a_{i-1} \). The interior of the triangle \( \triangle(a^*, a_{i-1}, a_j) \) is empty of points of \( Q \), since the triangle \( h \) is contained in \( h \). Moreover, \( \text{conv}(H) \) cannot intersect the line that determines \( h \) both strictly above and strictly below \( a^* \). Thus, all remaining points of \( H \setminus \{ a_{i-1} \} \) lie to the right of \( a_{i-1}a^* \) and to the right of \( a_ja^* \). If \( H \) is empty of points of \( P \setminus Q \), we are done. Otherwise, we let \( H' := (H \setminus \{ a_{i-1} \}) \cup \{ p' \} \) where \( p' \in P \setminus Q \) is a point inside \( \triangle(a^*, a_{i-1}, a_j) \) closest to \( a_ja^* \). Note that the point \( p' \) might not be unique. By construction, \( H' \) is an \( \ell \)-divided 5-hole in \( P \). An analogous argument shows that there is an \( \ell \)-divided 5-hole in \( P \) if \( H \) properly intersects \( a^*a_{i-1} \).

Finally, if \( H \) does not properly intersect any of the rays \( a^*a_{i-1} \) and \( a^*a_j \), then \( \text{conv}(H) \) contains no point of \( P \setminus Q \) in its interior, and hence \( H \) is an \( \ell \)-divided 5-hole in \( P \).

5.1 Sequences of \( a^* \)-wedges with at most two points of \( B \)

In this subsection we consider an \( \ell \)-divided set \( P = A \cup B \) with \( A \) not in convex position. We consider the union \( W \) of \( \ell \)-divided \( a^* \)-wedges, each containing at most two points of \( B \), and derive an upper bound on the number of points of \( B \) that lie in \( W \) if there is no \( \ell \)-divided 5-hole in \( P \cap W \); see Corollary 13.

**Observation 9.** Let \( P = A \cup B \) be an \( \ell \)-divided set with \( A \) not in convex position. Let \( W_k \) be an \( a^* \)-wedge with \( w_k \geq 1 \) and \( 1 \leq k \leq t \) and let \( b \) be the leftmost point in \( W_k \cap B \). Then the points \( a^*, a_{k-1}, b, \) and \( a_k \) form an \( \ell \)-divided 4-hole in \( P \).

From Observation 10 and Observation 9 we obtain the following result.

**Observation 10.** Let \( P = A \cup B \) be an \( \ell \)-divided set with \( A \) not in convex position and with no \( \ell \)-divided 5-hole in \( P \). Let \( W_k \) be an \( a^* \)-wedge with \( w_k \geq 2 \) and \( 1 \leq k \leq t \) and let \( b \) be the leftmost point in \( W_k \cap B \). For every point \( b' \) in \( (W_k \cap B) \setminus \{ b \} \), the line \( bb' \) intersects the segment \( a_k^{-1}a_k \). Consequently, \( b \) is inside \( \triangle(a_{k-1}, a_k, b') \), to the left of \( a_kb' \), and to the right of \( a_k^{-1}b' \).

The following lemma states that there is an \( \ell \)-divided 5-hole in \( P \) if two consecutive \( a^* \)-wedges both contain exactly two points of \( B \).

**Lemma 11.** Let \( P = A \cup B \) be an \( \ell \)-divided set with \( A \) not in convex position and with \( |A|, |B| \geq 5 \). Let \( W_i \) and \( W_{i+1} \) be consecutive \( a^* \)-wedges with \( w_i = 2 = w_{i+1} \) and \( 1 \leq i < t \). Then there is an \( \ell \)-divided 5-hole in \( P \).

**Proof.** Suppose for contradiction that there is no \( \ell \)-divided 5-hole in \( P \). Let \( W := W_i \cup W_{i+1} \) and let \( Q := P \cap W \). By Lemma 3 there is also no \( \ell \)-divided 5-hole in \( Q \). We label the points
in \( B \cap W_i \) as \( b_{i-1} \) and \( b_i \) so that \( b_{i-1} \) is to the right of \( b_i \). Similarly, we label the points in \( B \cap W_{i+1} \) as \( b_{i+1} \) and \( b_{i+2} \) so that \( b_{i+1} \) is to the right of \( b_{i+2} \). By Observation 10 the point \( a_i \) is to the right of \( b_i b_{i-1} \) and to the left of \( b_{i+1} b_{i+2} \). If the points \( b_{i-1}, b_i, b_{i+1}, b_{i+2} \) are in convex position, then \( a_i, b_{j+1}, b_{j+2}, b_{j-1}, b_j \) form an \( \ell \)-divided 5-hole in \( P \); see Figure 4(a). Thus, we assume the points \( b_{i-1}, b_i, b_{i+1}, b_{i+2} \) are not in convex position. Without loss of generality, we assume that \( b_i b_{i-1} \) intersects \( b_{i+1} b_{i+2} \).

We show that the segments \( a_i b_{i-1} \) and \( b_{i+1} b_{i+2} \) intersect. As \( b_i b_{i-1} \) intersects \( a_i a_{i-1} \) and \( b_{i+1} b_{i+2} \), the point \( b_{i-1} \) lies in the triangle \( \triangle(b_{i+1}, b_{i+2}, b_i) \). Moreover, \( b_{i-1} \) is to the right of \( b_{i+1} b_i \), \( a_i \) is to the left of \( b_{i+1} b_i \), \( b_i \) is to the left of \( a_i b_{i-1} \), and \( b_{i+1} \) is to the right of \( a_i b_{i-1} \). Consequently, the points \( a_i, b_{i+1}, b_{i-1}, b_i \) form an \( \ell \)-divided 4-hole in \( P \); see Figure 4(b).

**Figure 4:**

- (a) If \( b_{i-1}, b_i, b_{i+1}, b_{i+2} \) are in convex position, then there is an \( \ell \)-divided 5-hole in \( P \).
- (b) The points \( a^*, a_{i+1}, a_i, a_{i-1} \) form a 4-hole in \( P \).

The points \( a_{i-1}, b_i, b_{i-1}, b_{i+2} \) are in convex position because \( a_{i-1} \) is the leftmost and \( b_{i+2} \) is the rightmost of those four points and because both \( a_{i-1} \) and \( b_{i+2} \) lie to the left of \( b_i b_{i-1} \). Moreover, the points \( a_{i-1}, b_i, b_{i-1}, b_{i+2} \) form an \( \ell \)-divided 4-hole in \( P \) as \( \square(a_{i-1}, b_i, b_{i-1}, b_{i+2}) \) lies in \( W \) and \( w_i = w_{i+1} = 2 \).

We consider the four points \( b_{i+2}, b_{i-1}, b_{i+1}, a_{i+1} \). The point \( b_{i+2} \) is the rightmost of those four points. By Observation 10 \( b_{i+1} \) lies to the right of \( a_i b_{i+2} \) and \( a_{i+1} \) lies to the right of \( b_{i+1} b_{i+2} \). Since \( b_{i-1} \in W_i \) and \( b_{i+2} \in W_{i+1} \), the point \( b_{i-1} \) lies to the left of \( a_i b_{i+2} \). Thus, the clockwise order around \( b_{i+2} \) is \( a_{i+1}, b_{i+1}, b_{i-1} \).

Suppose for contradiction that the points \( b_{i+2}, b_{i-1}, b_{i+1}, a_{i+1} \) form a convex quadrilateral. Due to the clockwise order around \( b_{i+2} \), the convex quadrilateral is \( \square(b_{i+2}, b_{i-1}, b_{i+1}, a_{i+1}) \). The only points of \( P \) that can lie in the interior of this quadrilateral are \( a^*, a_{i-1}, a_i, \) and \( b_i \). Since the triangle \( \triangle(b_{i+2}, b_{i+1}, a_{i+1}) \) is contained in \( W_{i+1} \), it contains neither of the points \( a^*, a_{i-1}, a_i, \) and \( b_i \). Since the triangle \( \triangle(b_{i+2}, b_{i-1}, b_{i+1}) \) is contained in the convex hull of \( B \), it does not contain \( a^*, a_{i-1}, \) nor \( a_i \). Moreover, as \( b_{i-1} \) lies in the triangle \( \triangle(b_{i+2}, b_{i+1}, b_i) \), the triangle \( \triangle(b_{i+2}, b_{i-1}, b_{i+1}) \) also does not contain \( b_i \). Thus the quadrilateral \( \square(b_{i+2}, b_{i-1}, b_{i+1}, a_{i+1}) \) is empty of points of \( P \). By Observation 10, the two sectors \( S(a_{i-1}, b_i, b_{i-1}, b_{i+2}) \) and \( S(b_{i+2}, b_{i-1}, b_{i+1}, a_{i+1}) \) contain no point of \( P \). Since every point of \( B \setminus \{b_{i-1}, b_i, b_{i+1}, b_{i+2}\} \) is either in \( S(a_{i-1}, b_i, b_{i-1}, b_{i+2}) \) or in \( S(b_{i+2}, b_{i-1}, b_{i+1}, a_{i+1}) \), we have \( B = \{b_{i-1}, b_i, b_{i+1}, b_{i+2}\} \). This contradicts the assumption that \( |B| \geq 5 \).

Therefore the points \( b_{i+2}, b_{i-1}, b_{i+1}, a_{i+1} \) are not in convex position. In particular, the point \( b_{i+1} \) lies in the triangle \( \triangle(b_{i-1}, a_{i+1}, b_{i+2}) \), since \( a_{i+1} \) is the leftmost and \( b_{i+2} \) is the rightmost of the points \( b_{i+2}, b_{i-1}, b_{i+1}, a_{i+1} \) and since \( b_{i-1} \) lies in \( W_i \). The red area in Figure 4(b) gives an illustration.
Consequently, the point \( a_{i+1} \) lies to the left of \( b_{i+1}b_{i-1} \). By Observation 6(iii), the point \( a_{i+1} \) is not in the sector \( S(b_{i+1}, b_{i-1}, b_{i}, a_{i}) \), as otherwise the points \( b_{i+1}, b_{i-1}, b_{i}, a_{i}, a_{i+1} \) form an \( \ell \)-divided 5-hole in \( P \). Thus the point \( a_{i+1} \) lies to the left of \( \overrightarrow{a_ib_{i}} \); see Figure 4(b).

The points \( a^*, a_{i+1}, a_i, a_{i-1} \) do not form a 4-hole in \( P \) because otherwise \( b_i \) lies in the sector \( S(a_{i-1}, a^*, a_{i+1}, a_i) \), which is impossible by Observation 9(iii).

Therefore the points \( a^*, a_{i+1}, a_i, a_{i-1} \) are not in convex position. Now we show that \( a^* \) is inside the triangle \( \triangle(a_{i-1}, a_{i+1}, a_i) \). The point \( a_i \) is not inside \( \triangle(a_{i-1}, a_{i+1}, a^*) \), since, by Observation 7, \( a_i \) is to the right of \( a^* \) and since \( a^* \) is the rightmost inner point of \( A \). Since \( a_{i-1} \) is to the left of \( \overrightarrow{a^*a_i} \) and \( a_{i+1} \) is to the right of \( \overrightarrow{a^*a_i} \), \( a^* \) is the inner point of \( a^*, a_{i+1}, a_i, a_{i-1} \). Figure 5 gives an illustration.

\[
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{Location of the points of \( A \setminus Q \).}
\end{figure}
\]

Since \( |B| \geq 5 \), there is another \( a^* \)-wedge besides \( W_i \) and \( W_{i+1} \) that intersects \( \ell \). Now we show that all points of \( B \setminus Q \) lie in \( a^* \)-wedges below \( W_{i+1} \). The rays \( b_{i}a_{i-1} \) and \( b_{i-1}b_{i+2} \) both start in \( W_i \) and then leave \( W_i \). Moreover, the segment \( b_{i}a_{i-1} \) intersects \( \ell \) and \( b_{i-1}b_{i+2} \) intersects \( \overrightarrow{a^*a_i} \). As both \( b_{i} \) and \( b_{i-1} \) lie to the right of \( \overrightarrow{a_{i-1}b_{i+2}} \), all points of \( B \setminus Q \) that lie in an \( a^* \)-wedge above \( W_i \) also lie in the sector \( S(a_{i-1}, b_{i}, b_{i-1}, b_{i+2}) \). We recall that, by Observation 9(ii), the sector \( S(a_{i-1}, b_{i}, b_{i-1}, b_{i+2}) \) is empty of points of \( P \). Hence all points of \( B \setminus Q \) lie in \( a^* \)-wedges below \( W_{i+1} \).

We show that \( i = 1 \). That is, \( W_i \) is the topmost \( a^* \)-wedge that intersects \( \ell \). By Observation 7, \( a_{i+1} \) lies to the right of \( a^* \). Since \( a_i \) and \( a_{i+1} \) are both to the right of \( a^* \) and since \( a^* \) is inside the triangle \( \triangle(a_{i-1}, a_{i+1}, a_i) \), the point \( a_{i-1} \) is to the left of \( a^* \). By Observation 7, we have \( i = 1 \).

Now we show that all points of \( A \setminus Q \) lie to the left of \( \overrightarrow{a_{i+1}a_i} \), to right of \( \overrightarrow{a_{i+1}b_{i+1}} \), and to the right of \( \overrightarrow{a^*a_{i+1}} \). The violet area in Figure 5 gives an illustration where the remaining points of \( A \setminus Q \) lie. We recall that the sector \( S(a_{i-1}, b_{i}, b_{i-1}, b_{i+2}) \) (red shaded area in Figure 5) is empty of points of \( P \). By Observation 9, both sets \( \{a^*, a_i, b_i, a_{i-1}\} \) and \( \{a^*, a_{i+1}, b_{i+1}, a_i\} \) form \( \ell \)-divided 4-holes in \( P \). By Observation 9(ii), the two sectors \( S(a^*, a_i, b_i, a_{i-1}) \) (green shaded area in Figure 5) and \( S(a^*, a_{i+1}, b_{i+1}, a_i) \) (blue shaded area in Figure 5) are thus empty of points of \( P \). Therefore, no point of \( A \setminus Q \) lies to the left of \( \overrightarrow{a_{i+1}b_{i+1}} \). Since \( W \) is non-convex, every point of \( P \) that is to the left of \( \overrightarrow{a^*a_{i+1}} \) lies in \( Q \). Thus every point of \( A \setminus Q \) lies to the right of \( \overrightarrow{a^*a_{i+1}} \). Moreover, no point \( a \) of \( A \setminus Q \) lies to the right of \( \overrightarrow{a_{i+1}a_i} \) (gray area in Figure 5) because otherwise, \( a_{i+1} \) is an inner point of \( \triangle(a_i, a^*, a) \), which is impossible since \( a^* \) is the rightmost inner point of \( A \) and \( a_{i+1} \) is to the right of \( a^* \).
Now we have restricted where the points of \( A \setminus Q \) lie. In the rest of the proof we show that the points \( b_{i+2}, b_{i+1}, a_i, a_{i+1} \) form an \( \ell \)-divided 4-hole in \( P \). We will then use the sectors \( S(b_{i+2}, b_{i+1}, a_i, a_{i+1}) \) and \( S(a_{i-1}, b_i, b_{i-1}, b_{i+2}) \) to argue that \(|B| = |B \cap Q| = 4\), which then contradicts the assumption \(|B| \geq 5\).

We consider \( a_{i+2} \) and show that the points \( a_{i+1}, a^*, a_{i-1}, a_{i+2} \) are in convex position. It suffices to show that \( a_{i+2} \) does not lie in the triangle \( \triangle(a^*, a_{i-1}, a_{i+1}) \) because of the cyclic order of \( A \setminus \{a^*\} \) around \( a^* \). Recall that \( a^* \) lies inside the triangle \( \triangle(a_{i-1}, a_{i+1}, a_i) \), that \( b_{i+1} \) lies inside the triangle \( \triangle(a_i, a_{i+1}, b_{i+2}) \), and that \( b_{i-1} \) lies inside the triangle \( \triangle(a_{i-1}, a_{i+1}, b_{i+2}) \). Since the triangles \( \triangle(a_{i-1}, a_{i+1}, a_i), \triangle(a_i, a_{i+1}, b_{i+2}), \) and \( \triangle(a_{i-1}, a_i, b_{i+2}) \) are oriented counterclockwise along the boundary, the point \( a_i \) lies inside \( \triangle(a_{i-1}, a_{i+1}, b_{i+2}) \). Thus also the points \( a^*, b_i, b_{i+1} \) lie in the triangle \( \triangle(a_{i-1}, a_{i+1}, b_{i+2}) \). Consequently, the triangle \( \triangle(a^*, a_{i-1}, a_{i+1}) \) is contained in the union of the sectors \( S(a_{i+1}, b_{i+1}, a_i, a^*) \) (blue shaded area in Figure 5) and \( S(a^*, a_i, b_i, a_{i-1}) \) (green shaded area in Figure 5). Thus \( a_{i+2} \) does not lie in the triangle \( \triangle(a^*, a_{i-1}, a_{i+1}) \) and the points \( a_{i+1}, a^*, a_{i-1}, a_{i+2} \) are in convex position.

We now show that the sector \( S(a_{i+1}, a^*, a_{i-1}, a_{i+2}) \) is empty of points of \( P \). If the quadrilateral \( \square(a_{i+1}, a^*, a_{i-1}, a_{i+2}) \) is not empty of points of \( P \), then there is a point \( a'_{i-1} \) of \( A \) in \( \triangle(a^*, a_{i-1}, a_{i+2}) \). This is because \( \triangle(a^*, a_{i-2}, a_{i+1}) \) is empty of points of \( A \) due to the cyclic order of \( A \setminus \{a^*\} \) around \( a^* \). We can choose \( a'_{i-1} \) to be a point that is closest to the line \( a^*a_{i+2} \) among the points of \( A \) inside \( \triangle(a^*, a_{i+2}, a_{i+1}) \). If the quadrilateral \( \square(a_{i+1}, a^*, a_{i-1}, a_{i+2}) \) is empty of points of \( P \), then we set \( a'_{i-1} := a_{i-1} \).

By the choice of \( a'_{i-1} \), the quadrilateral \( \square(a_{i+1}, a^*, a'_{i-1}, a_{i+2}) \) is empty of points of \( P \). Since \( a_{i+1} \) and \( a_{i+2} \) are consecutive in the order around \( a^* \), no point of \( A \) lies in the sector \( S(a_{i+1}, a^*, a'_{i-1}, a_{i+2}) \). By Observation 6(ii), the sector \( S(a_{i+1}, a^*, a'_{i-1}, a_{i+2}) \) (gray shaded area in Figure 6(a)) is empty of points of \( P \). Since the sector \( S(a_{i+1}, a^*, a_{i-1}, a_{i+2}) \) is a subset of \( S(a_{i+1}, a^*, a'_{i-1}, a_{i+2}) \), the sector \( S(a_{i+1}, a^*, a_{i-1}, a_{i+2}) \) is empty of points of \( P \).

**Figure 6:** (a) Location of the points of \( B \setminus Q \). (b) The point \( a_{i+1} \) lies to the left of \( a_i \).

We show that \( a_{i+1} \) is to the left of \( a_i \) and to the right of \( a_{i+2} \). Recall that \( a_i \) lies to the right of \( a^* \) and to the left of \( b_i \). The point \( b_i \) lies to the left of \( a^*a_i \) and the point \( a_{i+1} \) lies to the right of this line; see Figure 6(b). The point \( a_{i+1} \) then lies to the left of \( a_i \), since we know already that \( a_{i+1} \) lies to the left of \( a^*b_i \). Recall that \( a_{i+1} \) is to the right of \( a^* \). Consequently,
the point \(a_{i+2}\) lies to the left of \(a_{i+1}\), as \(a_{i+2}\) lies to the right of \(a^*a_{i+1}\) and to the left of \(a_{i+1}\).

Now we are ready to prove that the points \(b_{i+2}, b_{i+1}, a_{i+1}, a_{i+2}\) form an \(\ell\)-divided 4-hole in \(P\) (green area in Figure 6(a)). Recall that \(b_{i+2}\) and \(a_{i+2}\) both lie to the right of \(a_{i+1}b_{i+1}\), and that \(a_{i+2}\) is the leftmost and \(b_{i+2}\) is the rightmost of those four points. Altogether, we see that the points \(b_{i+2}, b_{i+1}, a_{i+1}, a_{i+2}\) are in convex position. The four sectors \(S(b_{i+2}, a_{i-1}, b_i, b_{i-1})\) (red shaded area in Figure 6(a)), \(S(b_i, b_{i-1}, a_i, b_{i+1})\) (orange shaded area in Figure 6(a)), \(S(b_{i+1}, a_i, a^*, a_{i+1})\) (blue shaded area in Figure 6(a)), and \(S(a_{i+1}, a^*, a_{i+1}^*, a_{i+2})\) (gray shaded area in Figure 6(a)) contain the quadrilateral \(\square(b_{i+2}, b_{i+1}, a_{i+1}, a_{i+2})\) (green area in Figure 6(a)). The sectors are empty of points of \(P\) by Observation 6(i). Consequently, the convex quadrilateral \(\square(b_{i+2}, b_{i+1}, a_{i+1}, a_{i+2})\) is an \(\ell\)-divided 4-hole in \(P\).

To finish the proof, recall that all points of \(B \setminus Q\) lie in \(a^*\)-wedges below \(W_{i+1}\) as \(i = 1\). Since \(a_{i+2}\) is to the left of \(a_{i+1}\), the line \(\ell_{a_{i+2}a_{i+1}}\) intersects \(\ell\) above \(\ell\cap W_{i+2}\). The line \(\ell_{a_{i+1}b_{i+1}}\) also intersects \(\ell\) above \(\ell\cap W_{i+2}\), since \(a_{i+1}\) and \(b_{i+1}\) both lie in \(W_{i+1}\). From \(i = 1\), every point of \(B \setminus Q\) is to the right of \(a_{i+2}a_{i+1}\) and to the right of \(a_{i+1}b_{i+1}\). Since the points \(b_{i+2}, b_{i+1}, a_{i+1}, a_{i+2}\) form an \(\ell\)-divided 4-hole in \(P\), Observation 6(i) implies that the sector \(S(b_{i+2}, b_{i+1}, a_{i+1}, a_{i+2})\) is empty of points of \(P\). Thus every point of \(B \setminus Q\) lies to the left of \(b_{i+1}b_{i+2}\). Since \(b_{i+1}b_{i+2}\) intersects \(\ell\cap W_{i+1}\) above \(\ell\cap a_{i+1}b_{i+1}\) and since \(b_{i-1}\) lies to the left of \(b_{i+2}\) and to the left of \(b_{i+1}b_{i+2}\), every point of \(B \setminus Q\) lies to the left of \(b_{i-1}b_{i+2}\) and to the right of \(b_{i+2}\), and thus in the sector \(S(a_{i-1}, b_i, b_{i-1}, b_{i+2})\). However, by Observation 6(i), this sector is empty of points of \(P\). Thus we obtain \(B = \{b_{i-1}, b_i, b_{i+1}, b_{i+2}\}\), which contradicts the assumption \(|B| \geq 5\).

Next we show that if there is a sequence of consecutive \(a^*\)-wedges where the first and the last \(a^*\)-wedge both contain two points of \(B\) and every \(a^*\)-wedge in between them contains exactly one point of \(B\), then there is an \(\ell\)-divided 5-hole in \(P\).

**Lemma 12.** Let \(P = A \cup B\) be an \(\ell\)-divided set with \(A\) not in convex position and with \(|A| \geq 5\) and \(|B| \geq 6\). Let \(W_1, \ldots, W_j\) be consecutive \(a^*\)-wedges with \(1 \leq i < j \leq t\), \(w_i = 2 = w_j\), and \(w_k = 1\) for every \(k\) with \(i < k < j\). Then there is an \(\ell\)-divided 5-hole in \(P\).

**Proof.** For \(i = j - 1\), the statement follows by Lemma 11. Thus we assume \(j \geq i + 2\). That is, we have at least three consecutive \(a^*\)-wedges. Suppose for contradiction that there is no \(\ell\)-divided 5-hole in \(P\). Let \(W := \bigcup_{k=i}^{j} W_k\) and \(Q := P \cap W\). By Lemma 8, there is also no \(\ell\)-divided 5-hole in \(Q\). Note that \(|Q \cap B| = j - i + 3\). Also observe that \(|Q \cap A| = j - i + 2\) if \(a_{i-1} = a_j = a_i\) and \(|Q \cap A| = j - i + 3\) otherwise. We label the points in \(B \cap W_i\) as \(b_{i-1}\) and \(b_i\) so that \(b_{i-1}\) is to the right of \(b_i\). Further, we label the single point in \(B \cap W_k\) as \(b_k\) for each \(i < k < j\), and the two points in \(B \cap W_j\) as \(b_j\) and \(b_{j+1}\) so that \(b_{j+1}\) is to the right of \(b_j\); see Figure 7.

**Claim 12.1.** All points of \(B \cap (W_{k-1} \cup W_k \cup W_{k+1})\) are to the right of \(a_k a_{k-1}\) for every \(k\) with \(i < k < j\).

The claim clearly holds for points from \(B \cap W_k\). Thus it suffices to prove the claim only for points from \(B \cup W_{k-1}\), as for points from \(B \cup W_{k+1}\) it follows by symmetry. Since \(i < k < j\), Observation 7 implies that the points \(a_{k-1}\) and \(a_k\) are both to the right of \(a^*\).

We now distinguish the following two cases.
1. The point $a_{k-2}$ is to the left of $\overline{a^*a_k}$; see Figure 8(a). Since $a^*$ is the rightmost inner point of $A$, $a_{k-1}$ does not lie inside the triangle $\triangle(a^*, a_k, a_{k-2})$ and thus $\square(a_{k-2}, a^*, a_k, a_{k-1})$ is a 4-hole in $P$. All points of $B \cap W_{k-1}$ lie to the right of $\overline{a^*a_{k-2}}$ and to the left of $\overline{a_{k-2}a_{k-1}}$. By Observation 6(ii), no point of $B \cap W_{k-1}$ lies in the sector $S(a_{k-2}, a^*, a_k, a_{k-1})$ (red shaded area in Figure 5(a)) and thus all points of $B \cap W_{k-1}$ are to the right of $\overline{a^*a_{k-1}}$.

![Figure 7: An illustration of $a^*$-wedges $W_i, \ldots, W_j$ in the proof of Lemma 12](image)

2. The point $a_{k-2}$ is to the right of $\overline{a^*a_k}$; see Figure 8(b). Since $a_{k-1}$ and $a_k$ are to the right of $a^*$ and since $a_{k-2}$ is to the left of $\overline{a^*a_{k-1}}$ and to the right of $\overline{a^*a_k}$, the point $a_{k-2}$ is to the left of $a^*$. By Observation 7 we have $k = 2$. That is, $W_{k-1}$ is the topmost $a^*$-wedge that intersects $\ell$.

There is another $a^*$-wedge below $W_{k+1}$, since otherwise $|B| = |B \cap (W_{k-1} \cup W_k \cup W_{k+1})| \leq 2 + 1 + 2 = 5$, which is impossible according to the assumption $|B| \geq 6$. By Observation 7, the point $a_{k+1}$ is to the right of $a^*$. Moreover, since $a^*$ is the rightmost inner point of $A$, the point $a_k$ does not lie inside the triangle $\triangle(a^*, a_{k+1}, a_{k-1})$. The points $a^*, a_{k+1}, a_k, a_{k-1}$ then form a 4-hole in $P$, which has $a^*$ as the leftmost point.

By definition, all points of $B \cap W_{k-1}$ lie to the left of $\overline{a^*a_{k-1}}$. As the ray $\overrightarrow{a^*a_{k+1}}$ intersects $\ell$, all points of $B \cap W_{k-1}$ lie also to the left of $\overline{a^*a_{k+1}}$. By Observation 6(ii), no point of $B \cap W_{k-1}$ lies in the sector $S(a^*, a_{k+1}, a_k, a_{k-1})$. Thus all points of $B \cap W_{k-1}$ lie to the right of $\overline{a^*a_{k-1}}$.

![Figure 8: An illustration of the proof of Claim 12.1](image)
This finishes the proof of Claim 12.1.

We say that points \( p_1, p_2, p_3, p_4 \) form a **counterclockwise-oriented convex quadrilateral** if every triple \( (p_x, p_y, p_z) \) with \( 1 \leq x < y < z \leq 4 \) is oriented counterclockwise.

**Claim 12.2.** The points \( b_{i−1}, b_i, a_i, a_{i+1} \) form a counterclockwise-oriented convex quadrilateral.

Due to Claim 12.1 the points \( b_{i−1} \) and \( b_i \) are both to the right of \( \overrightarrow{a_i+1a_i} \). Thus the points \( a_i \) and \( a_{i+1} \) are both extremal points of those four points. Also the point \( b_{i−1} \) is extremal, since it is the rightmost of those four points. The point \( b_i \) does not lie inside the triangle \( \triangle(a_{i+1}, a_i, b_{i−1}) \), since, by Observation 10, \( b_i \) lies to the left of \( \overrightarrow{a_i b_{i−1}} \). To finish the proof of Claim 12.2, it suffices to observe that the triples \( (b_{i−1}, b_i, a_i) \), \( (b_{i−1}, b_i, a_{i+1}) \), \( (b_{i−1}, a_i, a_{i+1}) \), and \( (b_i, a_i, a_{i+1}) \) are all oriented counterclockwise.

**Claim 12.3.** The point \( b_{i+1} \) lies to the right of \( \overrightarrow{b_{i−1}b_i} \).

Suppose for contradiction that \( b_{i+1} \) lies to the left of \( \overrightarrow{b_{i−1}b_i} \). We consider the five points \( a_{i−1}, a_i, b_{i−1}, b_i, b_{i+1} \); see Figure 9. By Claim 12.1, the points \( b_{i−1}, b_i, b_{i+1} \) lie to the right of \( \overrightarrow{a_{i−1}a_i} \). Moreover, since \( b_{i−1} \) and \( b_i \) lie in \( W_i \) and since \( b_{i+1} \) lies in \( W_{i+1} \), the points \( b_{i−1} \) and \( b_i \) both lie to the left of \( \overrightarrow{a_i b_{i+1}} \). By Observation 10, the point \( a_{i−1} \) lies to the left of \( \overrightarrow{b_{i−1}b_i} \) and \( b_{i+1} \) is to the right of \( b_{i−1} \). Consequently, the points \( b_{i−1} \) and \( b_i \) lie in the triangle \( \triangle(a_{i−1}, a_i, b_{i+1}) \). Altogether, the points \( a_{i−1}, b_i, b_{i−1}, b_{i+1} \) are in convex position.

![Figure 9: An illustration of the proof of Claim 12.3](image)

By Claim 12.1 the points \( b_{i−1} \) and \( b_{i+1} \) lie to the right of \( \overrightarrow{a_{i+1}a_i} \). Moreover, since \( b_{i−1} \) is to the left of \( b_{i+1} \) and to the left of \( \overrightarrow{a_i b_{i+1}} \), the points \( b_{i−1}, b_i, a_i, a_{i+1} \) are in convex position. Since there are no further points in \( W_i \) and \( W_{i+1} \), the sets \( \{a_{i−1}, b_i, b_{i−1}, b_{i+1}\} \) and \( \{b_{i+1}, b_{i−1}, a_i, a_{i+1}\} \) are \( \ell \)-divided 4-holes in \( \mathcal{P} \). By Observation 10, the point \( b_{i+2} \) lies neither in \( S(a_{i−1}, b_i, b_{i−1}, b_{i+1}) \) nor in \( S(b_{i+1}, b_{i−1}, a_i, a_{i+1}) \). Recall that the ray \( \overrightarrow{b_{i−1}b_{i+1}} \) intersects \( a^∗a_i \) and the ray \( \overrightarrow{b_{i−1}a_i} \) does not intersect \( a^∗a_i \). Therefore \( b_{i+2} \) is to the right of \( \overrightarrow{a_i a_{i+1}} \). This contradicts Claim 12.1 and finishes the proof of Claim 12.3.

**Claim 12.4.** For each \( k \) with \( i < k < j \), the point \( b_k \) lies to the left of \( \overrightarrow{a_k b_{k−1}} \) and to the left of \( \overrightarrow{a_{k+1} b_{k−1}} \).

We show by induction on \( k \) that

(i) the points \( b_{k−1}, b_k, a_k, a_{k−1} \) and \( a_k b_{k−1} \) form a counterclockwise-oriented convex quadrilateral, which has \( b_{k−1} \) as the rightmost point, and

(ii) the point \( b_k \) lies inside this convex quadrilateral and, in particular, to the left of \( \overrightarrow{a_k b_{k−1}} \).
Claim \textbf{[12.4]} then clearly follows.

For the base case, we consider \( k = i + 1 \). By Claim \textbf{[12.2]}, the points \( b_{i-1}, b_i, a_i, \) and \( a_{i+1} \) form a counterclockwise-oriented convex quadrilateral. By definition, \( b_{i-1} \) is the rightmost of those four points. Figure \textbf{10}(a) gives an illustration. The point \( b_{i+1} \) lies to the right of \( a_{i+1}a_i \) and, by Claim \textbf{12.3}, to the right of \( b_i b_{i-1} \). Moreover, since \( b_{i+1} \) lies in \( W_{i+1} \), it lies to the right of \( a_{i}b_{i} \). By Observation \textbf{6}(b), \( b_{i+1} \) does not lie in the sector \( S(b_{i-1}, b_i, a_i, a_{i+1}) \). Consequently, \( b_{i+1} \) lies inside the quadrilateral \( \square(b_{i-1}, b_i, a_i, a_{i+1}) \).

![Figure 10: (a) An illustration of the proof of Claim 12.4. (b) An illustration of the proof of Lemma 12.](image)

For the inductive step, let \( i + 1 < k < j \). By the inductive assumption, the point \( b_{k-1} \) lies to the left of \( a_{k-1}b_{i-1} \) and to the left of \( b_{i-1} \). By Claim \textbf{12.1}, \( b_{k-1} \) lies to the right of \( a_{k}a_{k-1} \). Hence, the points \( a_{k} \) and \( b_{i-1} \) both lie to the right of \( a_{k-1}b_{i-1} \). Recall that the points \( b_{i-1}, b_{k-1}, a_{k-1}, a_{k} \) lie to the right of \( a^{*} \). Since \( b_{i-1} \) is the first and \( a_{k} \) is the last in the clockwise order around \( a^{*} \), the points \( b_{i-1}, b_{k-1}, a_{k-1}, a_{k} \) form a counterclockwise-oriented convex quadrilateral.

Recall that the points \( b_{k-1} \) and \( b_{k} \) both lie to the right of \( a_{k}a_{k-1} \) and that \( b_{k-1} \) is to the left of \( a_{k-1}b_{i-1} \). Since \( b_{k} \in W_{k} \), the point \( b_{k} \) lies to the right of \( a_{k-1}b_{i-1} \). Therefore the clockwise order of \( \{b_{i-1}, b_{k-1}, b_{k}\} \) around \( a_{k-1} \) is \( b_{k-1}, b_{k}, b_{i-1} \). Since \( b_{i-1} \) is not contained in \( W_{k-1} \cup W_{k} \), the point \( b_{i-1} \) is not contained in the triangle \( \triangle(a_{k-1}, b_{k}, b_{k-1}) \). Consequently, the points \( a_{k-1}, b_{k-1}, b_{k} \) form a convex quadrilateral and, in particular, \( b_{k} \) lies to the right of \( b_{k-1}b_{i-1} \). Figure \textbf{10}(a) gives an illustration. Since \( b_{k} \) lies in \( W_{k} \), it lies to the right of \( a_{k}b_{k-1} \).

By Observation \textbf{6}(b), the point \( b_{k} \) does not lie in the sector \( S(b_{i-1}, b_{k-1}, a_{k-1}, a_{k}) \). Thus \( b_{k} \) lies inside the quadrilateral \( \square(b_{i-1}, b_{k-1}, a_{k-1}, a_{k}) \). This finishes the proof of Claim \textbf{12.4}.

Using Claim \textbf{12.4}, we now finish the proof of Lemma \textbf{12} by finding an \( \ell \)-divided 5-hole in \( Q \) and thus obtaining a contradiction with the assumption that there is no \( \ell \)-divided 5-hole in \( P \). In the following, we assume, without loss of generality, that \( b_{j+1} \) is to the right of \( b_{i-1} \). Otherwise we can consider a vertical reflection of \( P \).

We consider the polygon \( \mathcal{P} \) through the points \( b_{i-1}, b_{j-1}, a_{j-1}, b_{j}, b_{j+1} \) and we show that \( \mathcal{P} \) is convex and empty of points of \( Q \). See Figure \textbf{10}(b) for an illustration. This will give us an \( \ell \)-divided 5-hole in \( Q \).

We show that \( \mathcal{P} \) is convex by proving that every point of \( \{b_{i-1}, b_{j-1}, a_{j-1}, b_{j}, b_{j+1}\} \) is a convex vertex of \( \mathcal{P} \). The point \( a_{j-1} \) is a convex vertex of \( \mathcal{P} \) because it is the leftmost point in \( \mathcal{P} \). The point \( b_{i-1} \) is a convex vertex of \( \mathcal{P} \) because all points of \( \mathcal{P} \) lie to the right of \( a^{*} \) and
$b_{i−1}$ is the topmost point in the clockwise order around $a^*$. The point $b_{j+1}$ is a convex vertex of $P$ because $b_{j+1}$ is the rightmost point in $P$ by Claim 12.4 and by the assumption that $b_{j+1}$ is to the right of $b_{i−1}$. The point $b_{j−1}$ is a convex vertex of $P$ because $b_{j−1}$ lies to the left of $a_{j−1}b_{i−1}$ by Claim 12.4 while $b_j$ and $b_{j+1}$ both lie to the right of this line. The point $b_i$ is a convex vertex of $P$ because, by Observation 10, $b_j$ lies to the right of $a_{j−1}b_{j+1}$ while $b_{j−1}$ and $b_{i−1}$ both lie to the right of this line. Consequently, $P$ is a convex pentagon with vertices from both $A$ and $B$. Moreover, by Claim 12.4, all points $b_k$ with $i < k < j$ lie to the left of $a_kb_{i−1}$. Since $b_i$ is to the left of $b_{j−1}b_{i−1}$, $P$ is thus empty of points of $Q$, which gives us a contradiction with the assumption that there is no $\ell$-divided 5-hole in $P$. \hfill \Box

We now use Lemma 12 to show the following upper bound on the total number of points of $B$ in a sequence $W_i, \ldots, W_j$ of consecutive $a^*$-wedges with $w_1, \ldots, w_j \leq 2$.

**Corollary 13.** Let $P = A \cup B$ be an $\ell$-divided set with no $\ell$-divided 5-hole, with $A$ not in convex position, and with $|A| \geq 5$ and $|B| \geq 6$. For $1 \leq i < j < t$, let $W_i, \ldots, W_j$ be consecutive $a^*$-wedges with $w_k \leq 2$ for every $k$ with $i \leq k \leq j$. Then $\sum_{k=i}^{j} w_k \leq j - i + 2$.

**Proof.** Let $n_0, n_1,$ and $n_2$ be the number of $a^*$-wedges from $W_i, \ldots, W_j$ with 0, 1, and 2 points of $B$, respectively. Due to Lemma 12, we can assume that between any two $a^*$-wedges from $W_i, \ldots, W_j$ with two points of $B$ each, there is an $a^*$-wedge with no point of $B$. Thus $n_2 \leq n_0 + 1$. Since $n_0 + n_1 + n_2 = j - i + 1$, we have $\sum_{k=i}^{j} w_k = 0n_0 + 1n_1 + 2n_2 = (j - i + 1) + (n_2 - n_0) \leq j - i + 2$. \hfill \Box

### 5.2 Computer-assisted results

We now provide lemmas that are key ingredients in the proof of Theorem 2. All these lemmas have computer-aided proofs. Each result was verified by two independent implementations, which are also based on different abstractions of point sets; see below for details.

**Lemma 14.** Let $P = A \cup B$ be an $\ell$-divided set with $|A| = 5$, $|B| = 6$, and with $A$ not in convex position. Then there is an $\ell$-divided 5-hole in $P$.

**Lemma 15.** Let $P = A \cup B$ be an $\ell$-divided set with no $\ell$-divided 5-hole in $P$, $|A| = 5$, $4 \leq |B| \leq 6$, and with $A$ in convex position. Then for every point $a$ of $A$, every convex $a$-wedge contains at most two points of $B$.

**Lemma 16.** Let $P = A \cup B$ be an $\ell$-divided set with no $\ell$-divided 5-hole in $P$, $|A| = 6$, and $|B| = 5$. Then for each point $a$ of $A$, every convex $a$-wedge contains at most two points of $B$.

**Lemma 17.** Let $P = A \cup B$ be an $\ell$-divided set with no $\ell$-divided 5-hole in $P$, $5 \leq |A| \leq 6$, $|B| = 4$, and with $A$ in convex position. Then for every point $a$ of $A$, if the non-convex $a$-wedge is empty of points of $B$, every $a$-wedge contains at most two points of $B$.

To prove these lemmas, we employ an exhaustive computer search through all combinatorially different sets of $|P| \leq 11$ points in the plane. Since none of these statements depends on the actual coordinates of the points but only on the relative positions of the points, we distinguish point sets only by orientations of triples of points as proposed by Goodman and Pollack [19]. That is, we check all possible equivalence classes of point sets in the plane with respect to their triple-orientations, which are known as order types.
We wrote two independent programs to verify Lemmas 14 to 17. Both programs are available online [25, 7].

The first implementation is based on programs from the two bachelor’s theses of Scheucher [26, 27]. For our verification purposes we reduced the framework from there to a very compact implementation [25]. The program uses the order type database [3, 6], which stores all order types realizable as point sets of size up to 11. The order types realizable as sets of ten points are available online [11] and the ones realizable as sets of eleven points need about 96 GB and are available upon request from Aichholzer. The running time of each of the programs in this implementation does not exceed two hours on a standard computer.

The second implementation [7] neither uses the order type database nor the program used to generate the database. Instead it relies on the description of point sets by so-called signature functions [8, 16]. In this description, points are sorted according to their x-coordinates and every unordered triple of points is represented by a sign from \{-, +\}, where the sign is \(-\) if the triple traced in the order by increasing x-coordinates is oriented clockwise and the sign is \(+\) otherwise. Every 4-tuple of points is then represented by four signs of its triples, which are ordered lexicographically. There are only eight 4-tuples of signs that we can obtain (out of 16 possible ones); see [8, Theorem 3.2] or [16, Theorem 7] for details. In our algorithm, we generate all possible signature functions using a simple depth-first search algorithm and verify the conditions from our lemmas for every signature. The running time of each of the programs in this implementation may take up to a few hundreds of hours.

5.3 Applications of the computer-assisted results

Here we present some applications of the computer-assisted results from Section 5.2.

Lemma 18. Let \(P = A \cup B\) be an \(\ell\)-divided set with no \(\ell\)-divided 5-hole in \(P\), with \(|A| \geq 6\), and with \(A\) not in convex position. Then the following two conditions are satisfied.

(i) Let \(W_i, W_{i+1}, W_{i+2}\) be three consecutive \(a^*\)-wedges whose union is convex and contains at least four points of \(B\). Then \(w_i, w_{i+1}, w_{i+2} \leq 2\).

(ii) Let \(W_i, W_{i+1}, W_{i+2}, W_{i+3}\) be four consecutive \(a^*\)-wedges whose union is convex and contains at least four points of \(B\). Then \(w_i, w_{i+1}, w_{i+2}, w_{i+3} \leq 2\).

Proof. To show part (i), let \(W := W_i \cup W_{i+1} \cup W_{i+2}, A' := A \cap W, B' := B \cap W\), and \(P' := A' \cup B'\). Since \(W\) is convex, \(P'\) is an island of \(P\) and thus there is no \(\ell\)-divided 5-hole in \(P'\). Note that \(|A'| = 5\) and \(A'\) is in convex position. If \(|B'| \leq 5\), then every convex \(a^*\)-wedge in \(P'\) contains at most two points of \(B'\) by Lemma 15 applied to \(P'\). So assume that \(|B'| \geq 6\). We remove points from \(P'\) from the right to obtain \(P'' = A' \cup B''\), where \(B''\) contains exactly six points of \(B'\). Note that there is no \(\ell\)-divided 5-hole in \(P''\), since \(P''\) is an island of \(P'\). By Lemma 15 each \(a^*\)-wedge in \(P''\) contains exactly two points of \(B''\). Let \(\hat{B}\) be the set of points of \(B\) that are to the left of the rightmost point of \(B''\), including this point, and let \(\hat{P} := A \cup \hat{B}\). Note that \(B'' \subseteq \hat{B}\). Since \(|B''| = 6\) and since \(W \cap \hat{B} = B''\), each of the \(a^*\)-wedges \(W_i, W_{i+1}, W_{i+2}\) contains exactly two points of \(\hat{B}\). The \(a^*\)-wedges \(W_i, W_{i+1}, W_{i+2}\) are also \(a^*\)-wedges in \(\hat{P}\). Thus, Lemma 11 applied to \(\hat{P}\) and \(W_i, W_{i+1}\) then gives us an \(\ell\)-divided 5-hole in \(\hat{P}\). From the choice of \(\hat{P}\), we then have an \(\ell\)-divided 5-hole in \(P\), a contradiction.

To show part (ii), let \(W := W_i \cup W_{i+1} \cup W_{i+2} \cup W_{i+3}, A' := A \cap W, B' := B \cap W\), and \(P' := A' \cup B'\). Since \(W\) is convex, \(P'\) is an island of \(P\) and thus there is no \(\ell\)-divided 5-hole in \(P''\). Note that \(|A'| = 6\) and \(A'\) is in convex position. If \(|B'| = 4\), then the statement
follows from Lemma 17 applied to $P'$ since $a^*$ is an extremal point of $P'$. If $|B'| = 5$, then the statement follows from Lemma 16 applied to $P'$ and thus we can assume $|B'| \geq 6$. Suppose for contradiction that $w_j \geq 3$ for some $i \leq j \leq i + 3$. We remove points from $P$ from the right to obtain $P''$ so that $B'' := P'' \cap B$ contains exactly six points of $W \cap B$. By applying part (i) for $P''$ and $W_i \cup W_{i+1} \cup W_{i+2}$ and $W_{i+1} \cup W_{i+2} \cup W_{i+3}$, we obtain that $|B'' \cap W_i|, |B'' \cap W_{i+1}|, |B'' \cap W_{i+2}| = 0$. Let $b$ be the rightmost point from $P'' \cap W$. By Lemma 16 applied to $W \cap (P'' \setminus \{b\})$, there are at most two points of $B'' \setminus \{b\}$ in every $a^*$-wedge in $W \cap (P'' \setminus \{b\})$. This contradicts the fact that either $|(B'' \cap W_i) \setminus \{b\}| = 3$ or $|(B'' \cap W_{i+3}) \setminus \{b\}| = 3$.

\[\square\]

5.4 Extremal points of $\ell$-critical sets

Recall the definition of $\ell$-critical sets: An $\ell$-divided point set $C = A \cup B$ is called $\ell$-critical if neither $C \cap A$ nor $C \cap B$ is in convex position and if for every extremal point $x$ of $C$, one of the sets $(C \setminus \{x\}) \cap A$ and $(C \setminus \{x\}) \cap B$ is in convex position.

In this section, we consider an $\ell$-critical set $C = A \cup B$ with $|A|, |B| \geq 5$. We first show that $C$ has at most two extremal points in $A$ and at most two extremal points in $B$. Later, under the assumption that there is no $\ell$-divided 5-hole in $C$, we show that $|B| \leq |A| - 1$ if $A$ contains two extremal points of $C$ (Section 5.4.1) and that $|B| \leq |A|$ if $B$ contains two extremal points of $C$ (Section 5.4.2).

**Lemma 19.** Let $C = A \cup B$ be an $\ell$-critical set. Then the following statements are true.

(i) If $|A| \geq 5$, then $|A \cap \partial \text{conv}(C)| \leq 2$.

(ii) If $A \cap \partial \text{conv}(C) = \{a, a'\}$, then $a^*$ is the single interior point in $A$ and every point of $A \setminus \{a, a'\}$ lies in the convex region spanned by the lines $a^*a$ and $a^*a'$ that does not have any of $a$ and $a'$ on its boundary.

(iii) If $A \cap \partial \text{conv}(C) = \{a, a'\}$, then the $a^*$-wedge that contains $a$ and $a'$ contains no point of $B$.

By symmetry, analogous statements hold for $B$.

**Proof.** To show statement (i), suppose for contradiction that $|A \cap \partial \text{conv}(C)| \geq 3$. Let $a, a'$, and $a''$ be three such consecutive points. If there is no point of $A$ in the triangle $\triangle(a, a', a'')$ spanned by the points $a, a'$, and $a''$, then $A \setminus \{a'\}$ is not in convex position. This is impossible, since $C$ is an $\ell$-critical set. If there is at least one point $a^{(1)}$ in $\triangle(a, a', a'')$, then we consider an arbitrary point $a^{(2)}$ from $A \setminus \{a', a'', a^{(1)}\}$. Such a point $a^{(2)}$ exists, since $|A| \geq 5$. The point $a^{(1)}$ lies inside one of the triangles $\triangle(a, a', a^{(2)})$, $\triangle(a, a'', a^{(2)})$, or in $\triangle(a', a'', a^{(2)})$ and thus one of the sets $A \setminus \{a''\}$, $A \setminus \{a'\}$, or $A \setminus \{a\}$ is not in convex position, which is again impossible. In any case, $C$ cannot be $\ell$-critical and we obtain a contradiction.

To show statement (ii), assume that $A \cap \partial \text{conv}(C) = \{a, a'\}$. Every triangle in $A$ with a point of $A$ in its interior has $a$ and $a'$ as vertices, as otherwise $A \setminus \{a\}$ or $A \setminus \{a'\}$ is not in convex position, which is impossible. Consider points $a^{(1)}$ and $a^{(2)}$ from $A$ such that $\triangle(a, a', a^{(1)})$ contains $a^{(2)}$. Denote by $R$ the region bounded by $aa^{(2)}$ and $a^{(2)}a^{(1)}$ that contains $a^{(1)}$. If there is a point $a^{(3)}$ in $A \setminus (R \cup \{a, a'\})$ then $a^{(2)}$ lies in one of $\triangle(a, a^{(1)}, a^{(3)})$ and $\triangle(a', a^{(1)}, a^{(3)})$, implying that $A \setminus \{a\}$ or $A \setminus \{a'\}$ is not in convex position. Hence all points of $A \setminus \{a, a', a^{(2)}\}$ lie in $R$. Moreover, any further interior point $a^{(4)}$ from $A \cap R$ lies in some triangle $\triangle(a, a', a^{(5)})$ for
some $a^{(5)} \in A \cap R$. Thus, $a^{(4)}$ also lies in one of the triangles $\triangle(a,a^{(2)},a^{(5)})$ or $\triangle(a',a^{(2)},a^{(5)})$. This implies that $A \setminus \{a\}$ or $A \setminus \{a'\}$ is not in convex position. Hence $a^{(2)}$ is the only interior point of $A$.

To show statement (iii), assume that $A \cap \partial \text{conv}(C) = \{a,a'\}$. Let $W_i$ be the wedge that contains $a$ and $a'$. Since $a$ and $a'$ are the only extremal points of $C$ contained in $A$, the segment $aa'$ is an edge of $\text{conv}(C)$. The points $a$, $a'$, and $a''$ all lie in $A$ and thus the triangle $\triangle(a,a',a'')$ contains no points of $B$. Since all points of $C$ lie in the closed halfplane that is determined by the line $\overline{aa''}$ and that contains $a''$, the wedge $W_i$ contains no points of $B$. \hfill $\square$

We remark that the assumption $|A| \geq 5$ in part (i) of Lemma 19 is necessary. In fact, arbitrarily large $\ell$-critical sets with only four points in $A$ and with three points of $A$ on $\partial \text{conv}(C)$ exist, and analogously for $B$. Figure 2(c) gives an illustration.

**Lemma 20.** Let $C = A \cup B$ be an $\ell$-critical set with no $\ell$-divided 5-hole in $C$ and with $|A| \geq 6$. Then $w_i \leq 3$ for every $1 < i < t$. Moreover, if $|A \cap \partial \text{conv}(C)| = 2$, then $w_1, w_t \leq 3$.

**Proof.** Recall that, since $C$ is $\ell$-critical, we have $|B| \geq 4$. Let $i$ be an integer with $1 \leq i \leq t$. We assume that there is a point $a$ in $A \cap \partial \text{conv}(C)$, which lies outside of $W_i$, as otherwise there is nothing to prove for $W_i$ (either $|A \cap \partial \text{conv}(C)| = 1$ and $i \in \{1,t\}$ or $|A \cap \partial \text{conv}(C)| = 2$ and, by Lemma 19(iii), $W_i \cap B = \emptyset$). We consider $C' := C \setminus \{a\}$. Since $C$ is an $\ell$-critical set, $A' := C' \cap A$ is in convex position. Thus, there is a non-convex $a^*$-wedge $W'$ of $C'$. Since $W'$ is non-convex, all other $a^*$-wedges of $C'$ are convex. Moreover, since $W'$ is the union of the two $a^*$-wedges of $C$ that contain $a$, all other $a^*$-wedges of $C'$ are also $a^*$-wedges of $C$. Let $W$ be the union of all $a^*$-wedges of $C$ that are not contained in $W'$. Note that $W$ is convex and contains at least $|A| - 3 \geq 3 a^*$-wedges of $C$. Since $|A| \geq 6$, the statement follows from Lemma 18(iii). \hfill $\square$

### 5.4.1 Two extremal points of $C$ in $A$

**Proposition 21.** Let $C = A \cup B$ be an $\ell$-critical set with no $\ell$-divided 5-hole in $C$, with $|A|, |B| \geq 6$, and with $|A \cap \partial \text{conv}(C)| = 2$. Then $|B| \leq |A| - 1$.

**Proof.** Since $|A \cap \partial \text{conv}(C)| = 2$, Lemma 20 implies that $w_i \leq 3$ for every $1 \leq i \leq t$. Let $a$ and $a'$ be the two points in $A \cap \partial \text{conv}(C)$. By Lemma 19(iii), all points of $A \setminus \{a,a'\}$ lie in the convex region $R$ spanned by the lines $\overline{aa}$ and $\overline{a'a}$ that do not have any of $a$ and $a'$ on its boundary. That is, without loss of generality, $a = a_{h-1}$ and $a' = a_h$ for some $1 \leq h \leq |A| - 1$ and, by Lemma 19(iii), we have $w_h = 0$. Since all points of $A \setminus \{a,a'\}$ lie in the convex region $R$, the regions $W := \text{cl}(\mathbb{R}^2 \setminus (W_{h-1} \cup W_h))$ and $W' := \text{cl}(\mathbb{R}^2 \setminus (W_h \cup W_{h+1}))$ are convex. Here $\text{cl}(X)$ denotes the closure of a set $X \subseteq \mathbb{R}^2$. Recall that the indices of the $a^*$-wedges are considered modulo $|A| - 1$ and that $\mathbb{R}^2$ is the union of all $a^*$-wedges.

First, suppose for contradiction that $|A| = 6$ and $|B| \geq 6$. There are exactly five $a^*$-wedges $W_1, \ldots, W_5$, and only four of them can contain points of $B$, since $w_0 = 0$. We apply Lemma 18 to $W$ and to $W'$ and obtain that either $w_i \leq 2$ for every $1 \leq i \leq t$ or $w_{h-1}, w_{h+1} = 3$ and $w_i = 0$ for every $i \notin \{h-1,h+1\}$. In the first case, Corollary 13 implies that $|B| \leq 5$ and in the latter case Lemma 16 applied to $P \setminus \{b\}$, where $b$ is the rightmost point of $B$, gives $|B| \leq 5$, a contradiction. Hence, we assume $|A| \geq 7$.

**Claim 21.1.** For $1 \leq k \leq t - 3$, if one of the four consecutive $a^*$-wedges $W_k, W_{k+1}, W_{k+2}$, or $W_{k+3}$ contains 3 points of $B$, then $w_k + w_{k+1} + w_{k+2} + w_{k+3} = 3$. 

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There are $|A| - 1 \geq 6$ $a^*$-wedges and, in particular, $W$ and $W'$ are both unions of at least four $a^*$-wedges. For every $W_i$ with $w_i = 3$ and $1 \leq i \leq t$, the $a^*$-wedge $W_i$ is either contained in $W$ or in $W'$. Thus we can find four consecutive $a^*$-wedges $W_k, W_{k+1}, W_{k+2}, W_{k+3}$ whose union is convex and contains $W_i$. Lemma [18.3] implies that each of $W_k, W_{k+1}, W_{k+2}, W_{k+3}$ except of $W_i$ is empty of points of $B$. This finishes the proof of Claim 21.1.

Claim 21.2. For all integers $i$ and $j$ with $1 \leq i < j \leq t$, we have $\sum_{k=i}^{j} w_k \leq j - i + 2$.

Let $S := (w_1, \ldots, w_j)$ and let $S'$ be the subsequence of $S$ obtained by removing every 1-entry from $S$. If $S$ contains only 1-entries, the statement clearly follows. Thus we can assume that $S'$ is non-empty. Recall that $S'$ contains only 0-, 2-, and 3-entries, since $w_i \leq 3$ for all $1 \leq i \leq t$. Due to Claim 21.1, there are at least three consecutive 0-entries between every pair of nonzero entries of $S'$ that contains a 3-entry. Together with Lemma 12 this implies that there is at least one 0-entry between every pair of 2-entries in $S'$.

By applying the following iterative procedure, we show that $\sum_{s \in S'} s \leq |S'| + 1$. While there are at least two nonzero entries in $S'$, we remove the first nonzero entry $s$ from $S'$. If $s = 2$, then we also remove the 0-entry from $S'$ that succeeds $s$ in $S$. If $s = 3$, then we also remove the two consecutive 0-entries from $S'$ that succeed $s$ in $S'$. The procedure stops when there is at most one nonzero element $s'$ in the remaining subsequence $S''$ of $S'$. If $s' = 3$, then $S''$ contains at least one 0-entry and thus $S''$ contains at least $s' - 1$ elements. Since the number of removed elements equals the sum of the removed elements in every step of the procedure, we have $\sum_{s \in S'} s \leq |S'| + 1$. This implies

$$\sum_{k=i}^{j} w_k = \sum_{s \in S} s = |S| - |S'| + \sum_{s \in S'} s \leq |S| - |S'| + |S'| + 1 = j - i + 2$$

and finishes the proof of Claim 21.2.

If $W_h$ does not intersect $\ell$, that is, $t < h \leq |A| - 1$, then the statement follows from Claim 21.2 applied with $i = 1$ and $j = t$. Otherwise, we have $h = 1$ or $h = t$ and we apply Claim 21.2 with $(i, j) = (2, t)$ or $(i, j) = (1, t - 1)$, respectively. Since $t \leq |A| - 1$ and $w_h = 0$, this gives us $|B| \leq |A| - 1$.

5.4.2 Two extremal points of $C$ in $B$

Proposition 22. Let $C = A \cup B$ be an $\ell$-critical set with no $\ell$-divided 5-hole in $C$, with $|A|, |B| \geq 6$, and with $|B \cap \partial \text{conv}(C)| = 2$. Then $|B| \leq |A|$.

Proof. If $w_k \leq 2$ for all $1 \leq k \leq t$, then the statement follows from Corollary [13] since $|B| = \sum_{k=1}^{t} w_k \leq t + 1 \leq |A|$. Therefore we assume that there is an $a^*$-wedge $W_i$ that contains at least three points of $B$. Let $b_1, b_2$, and $b_3$ be the three leftmost points in $W_i \cap B$ from left to right. Without loss of generality, we assume that $b_3$ is to the left of $b_1b_2$. Otherwise we can consider a vertical reflection of $P$. Figure [11] gives an illustration.

Let $R_1$ be the region that lies to the left of $b_1b_2$ and to the right of $b_2b_3$ and let $R_2$ be the region that lies to the right of $a_2a_3$ and to the right of $a_3a_4$. Let $B' := B \setminus \{b_1, b_2, b_3\}$.

Claim 22.1. Every point of $B'$ lies in $R_1 \cup R_2$.

We first show that every point of $B'$ that lies to the left of $b_1b_2$ lies in $R_1$. Then we show that every point of $B'$ that lies to the right of $b_1b_2$ lies in $R_2$. 

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By Observation 10, both lines \( \overline{b_1b_2} \) and \( \overline{b_1b_3} \) intersect the segment \( a_{i-1}a_i \). Since the segment \( a_{i-1}b_i \) intersects \( \ell \) and since \( b_1 \) is the leftmost point of \( W_i \cap B \), all points of \( B' \) that lie to the left of \( \overline{b_1b_2} \) lie to the left of \( \overline{a_{i-1}b_1} \). The four points \( a_{i-1}, b_1, b_2, b_3 \) form an \( \ell \)-divided 4-hole in \( P \), since \( a_{i-1} \) is the leftmost and \( b_3 \) is the rightmost point of \( a_{i-1}, b_1, b_2, b_3 \) and both \( a_{i-1} \) and \( b_3 \) lie to the left of \( \overline{b_1b_2} \). By Observation 5, the sector \( S(a_{i-1}, b_1, b_2, b_3) \) is empty of points of \( P \) (green shaded area in Figure 11). Altogether, all points of \( B' \) that lie to the left of \( b_1b_2 \) are to the right of \( b_1b_3 \) and thus lie in \( R_1 \).

Since the segment \( a_i b_1 \) intersects \( \ell \) and since \( b_1 \) is the leftmost point of \( W_i \cap B \), all points of \( B' \) that lie to the right of \( b_1b_2 \) lie to the right of \( a_i b_1 \). By Observation 6, the sector \( S(b_1, b_2, b_3, a_{i-1}) \) is empty of points of \( P \). Combining this with the fact that \( a^* \) is to the right of \( a_{i-1}b_3 \), we see that \( a^* \) lies to the right of \( \overline{b_1b_2} \). Since \( b_1 \) and \( b_2 \) both lie to the left of \( \overline{a^*a_i} \) and since \( a^* \) and \( a_i \) both lie to the right of \( b_1b_2 \), the points \( b_2, b_3, a^*, a_i \) form an \( \ell \)-divided 4-hole in \( P \). By Observation 3, the sector \( S(b_2, b_1, a^*, a_i) \) (blue shaded area in Figure 11) is empty of points of \( P \). Altogether, all points of \( B' \) that lie to the right of \( b_1b_2 \) are in convex position. Due to Observation 10, the sector \( S(b_1, a_{i-1}, b_1, a_i) \) is empty of points from \( P \). Thus \( b_4 \) lies to the right of \( b_2b_3 \) and the statement follows.

Second, we assume that the points \( b_2, b_3, b_1, a_i \) are not in convex position. Due to Observation 10, \( b_2 \) and \( b_3 \) both lie to the right of \( \overline{a_i b_1} \). Moreover, since \( b_3 \) is the rightmost of those four points, \( b_2 \) lies inside the triangle \( \triangle(b_3, b_1, a_i) \). In particular, \( a_i \) lies to the right of \( \overline{b_2b_3} \). Therefore, since \( b_2 \) and \( b_3 \) are to the left of \( \overline{a^*a_i} \), the line \( \overline{b_2b_3} \) intersects \( \ell \) at point \( p \). Let \( q \) be the point \( \ell \cap \overline{b_2b_3} \). Note that \( q \) is to the left of \( \overline{a^*a_i} \). The point \( b_4 \) is to the right of \( \overline{b_2b_3} \), as otherwise \( b_4 \) lies in \( \triangle(p, q, b_2) \), which is impossible because the points \( p, q, b_2 \) are in \( W_i \) while \( b_4 \) is not. Altogether, \( b_2 \) is inside \( \triangle(b_3, b_1, b_4) \) and this finishes the proof of Claim 22.2.

**Claim 22.3.** Either every point of \( B' \) is to the right of \( b_3 \) or \( b_3 \) is the rightmost point of \( B \).

By Observation 6, the sector \( S(b_3, a_{i-1}, b_1, b_2) \) is empty of points of \( P \) and thus all points of \( B' \cap R_1 \) lie to the left of \( \overline{a_{i-1}b_3} \) and, in particular, to the right of \( b_3 \).
Suppose for contradiction that the claim is not true. That is, there is a point $b_4 \in B'$ that is the rightmost point in $B$ and there is a point $b_5 \in B'$ that is to the left of $b_3$. Note that $b_4$ is an extremal point of $C$. By Claim 22.1 and by the fact that all points of $B' \cap R_1$ lie to the right of $b_3$, $b_5$ lies in $R_2 \setminus R_1$. By Claim 22.2 $b_2$ lies in the triangle $\triangle(b_1, b_5, b_3)$, and thus $B \setminus \{b_4\}$ is not in convex position. This contradicts the assumption that $C$ is an $\ell$-critical island. This finishes the proof of Claim 22.3.

Claim 22.4. The point $b_3$ is the third leftmost point of $B$. In particular, $W_i$ is the only $a^*$-wedge with at least three points of $B$.

Suppose for contradiction that $b_3$ is not the third leftmost point of $B$. Then by Claim 22.3 $b_3$ is the rightmost point of $B$ and therefore an extremal point of $B$. This implies that $B' \subseteq R_2 \setminus R_1$, since all points of $B' \cap R_1$ lie to the right of $b_3$. By Claim 22.2 each point of $B'$ then forms a non-convex quadrilateral together with $b_1$, $b_2$, and $b_3$. Since neither $b_1$ nor $b_2$ are extremal points of $C$ and since $|B \cap \partial \text{conv}(C)| = 2$, there is a point $b_4 \in B$ that is an extremal point of $C$. Since $|B| \geq 5$, the set $C \setminus \{b_4\}$ has none of its parts separated by $\ell$ in convex position, which contradicts the assumption that $C$ is an $\ell$-critical set. Since $W_i$ is an arbitrary $a^*$-wedge with $w_i \geq 3$, Claim 22.4 follows.

Claim 22.5. Let $W$ be a union of four consecutive $a^*$-wedges that contains $W_i$. Then $|W \cap B| \leq 4$.

Suppose for contradiction that $|W \cap B| \geq 5$. Let $C' \coloneqq C \cap W$. Note that $|C' \cap A| = 6$ and that $a^*, a_{i-1}, a_i$ lie in $C'$. By Lemma 5 there is no $\ell$-divided 5-hole in $C'$. We obtain $C''$ by removing points from $C'$ from the right until $|C'' \cap B| = 5$. Since $C''$ is an island of $C'$, there is no $\ell$-divided 5-hole in $C''$. From Claim 22.4 we know that $b_1, b_2, b_3$ are the three leftmost points in $C$ and thus lie in $C''$. We apply Lemma 16 to $C''$ and, since $b_1, b_2, b_3$ lie in a convex $a^*$-wedge of $C''$, we obtain a contradiction. This finishes the proof of Claim 22.5.

We now complete the proof of Proposition 22. First, we assume that $1 \leq i \leq 4$. Let $W \coloneqq W_1 \cup W_2 \cup W_3 \cup W_4$. By Claim 22.5 $|W \cap B| \leq 4$. Claim 22.4 implies that $w_k \leq 2$ for every $k$ with $5 \leq k \leq t$. By Corollary 13 we have

$$|B| = \sum_{k=1}^{4} w_k + \sum_{k=5}^{t} w_k \leq 4 + (t - 3) = t + 1 \leq |A|.$$  

The case $t - 3 \leq i \leq t$ follows by symmetry.

Second, we assume that $5 \leq i \leq t - 4$. Let $W \coloneqq W_{i-3} \cup W_{i-2} \cup W_{i-1} \cup W_i$. Note that $W$ is convex, since $2 \leq i - 3$ and $i < t$. By Lemma 18, we have $w_{i-3} + w_{i-2} + w_{i-1} + w_i \leq 3$ and $w_i + w_{i+1} + w_{i+2} + w_{i+3} \leq 3$. By Claim 22.4 $w_k \leq 2$ for all $k$ with $1 \leq k \leq i - 4$. Thus, by Corollary 13 $\sum_{k=1}^{i-4} w_k \leq i - 3$. Similarly, we have $\sum_{k=i+4}^{t} w_k \leq t - i - 2$. Altogether, we obtain that

$$|B| = \sum_{k=1}^{i-4} w_k + \sum_{k=i-3}^{i-1} w_k + w_i + \sum_{k=i+1}^{i+3} w_k + \sum_{k=i+4}^{t} w_k \leq (i - 3) + 3 + (t - i - 2) = t - 2 \leq |A| - 3.$$  

\qed
5.5 Finalizing the proof of Theorem 2

We are now ready to prove Theorem 2. Namely, we show that for every \( \ell \)-divided set \( P = A \cup B \) with \( |A|, |B| \geq 5 \) and with neither \( A \) nor \( B \) in convex position there is an \( \ell \)-divided 5-hole in \( P \).

Suppose for the sake of contradiction that there is no \( \ell \)-divided 5-hole in \( P \). By the result of Harborth [20], every set \( P \) of ten points contains a 5-hole in \( P \). In the case \( |A|, |B| = 5 \), the statement then follows from the assumption that neither of \( A \) and \( B \) is in convex position.

So assume that at least one of the sets \( A \) and \( B \) has at least six points. We obtain an island \( Q \) of \( P \) by iteratively removing extremal points so that neither part is in convex position after the removal and until one of the following conditions holds.

(i) One of the parts \( Q \cap A \) and \( Q \cap B \) has only five points.

(ii) \( Q \) is an \( \ell \)-critical island of \( P \) with \( |Q \cap A|, |Q \cap B| \geq 6 \).

In case (i), we have \( |Q \cap A| = 5 \) or \( |Q \cap B| = 5 \). If \( |Q \cap A| = 5 \) and \( |Q \cap B| \geq 6 \), then we let \( Q' \) be the union of \( Q \cap A \) with the six leftmost points of \( Q \cap B \). Since \( Q \cap A \) is not in convex position, Lemma 14 implies that there is an \( \ell \)-divided 5-hole in \( Q' \), which is also an \( \ell \)-divided 5-hole in \( Q \), since \( Q' \) is an island of \( Q \). However, this is impossible as then there is an \( \ell \)-divided 5-hole in \( P \) because \( Q \) is an island of \( P \). If \( |Q \cap A| \geq 6 \) and \( |Q \cap B| = 5 \), then we proceed analogously.

In case (ii), we have \( |Q \cap A|, |Q \cap B| \geq 6 \). There is no \( \ell \)-divided 5-hole in \( Q \), since \( Q \) is an island of \( P \). By Lemma 19, we can assume without loss of generality that \( |A \cap \partial \text{conv}(Q)| = 2 \). Then it follows from Proposition 21 that \( |Q \cap B| < |Q \cap A| \). By exchanging the roles of \( Q \cap A \) and \( Q \cap B \) and by applying Proposition 22, we obtain that \( |Q \cap A| \leq |Q \cap B| \), a contradiction. This finishes the proof of Theorem 2.

6 Final Remarks

At a first glance, it might seem that a similar approach could be used to derive stronger lower bounds also on the minimum number of 6-holes \( h_6(n) \). However, since there are point sets of 29 points with no 6-hole [23], one would need to investigate point sets of size at least 30 in order to find an \( \ell \)-divided 6-hole. This task is too demanding for our implementations, since the number of combinatorially different point sets grows too rapidly. Moreover, the case analysis in several steps of our proof would become much more complicated.

6.1 Necessity of the assumptions in Theorem 2

In the statement of Theorem 2, we require that the \( \ell \)-divided set \( P = A \cup B \) satisfies \( |A|, |B| \geq 5 \). We now show that those requirements are necessary in order to guarantee an \( \ell \)-divided 5-hole in \( P \) by constructing an arbitrarily large \( \ell \)-critical set \( C = A \cup B \) with \( |A| = 4 \) and with no \( \ell \)-divided 5-hole in \( C \).

Proposition 23. For every integer \( n \geq 5 \), there exists an \( \ell \)-critical set \( C = A \cup B \) with \( |A| = 4, |B| = n \), and with no \( \ell \)-divided 5-hole in \( C \).
Proof. First, we consider the case where \( n \) is odd. Let \( p^+ = (0,1) \) and \( p^- = (0,-1) \) be two auxiliary points and let \( \ell^+ = \{(x,y) \in \mathbb{R}^2 : y = x/4\} \) and \( \ell^- = \{(x,y) \in \mathbb{R}^2 : y = -x/4\} \) be two auxiliary lines. We place the point \( b'_i = (2, -1/2) \) on the line \( \ell^- \) and the auxiliary point \( q = (2, 1/2) \) on the line \( \ell^+ \). For \( i = 2, \ldots, n \), we iteratively let \( b'_i \) be the intersection of the line \( \ell^+ \) with the segment \( p^+ b'_{i-1} \) if \( i \) is even and the intersection of \( \ell^- \) with \( p^- b'_{i-1} \) if \( i \) is odd.

We place two points \( a_1 \) and \( a_2 \) sufficiently close to \( p^+ \) so that \( a_1 \) is above \( a_2 \), the segment \( a_1 a_2 \) is vertical with the midpoint \( p^+ \), and all non-collinear triples \( (b'_i, b'_j, p^+) \) have the same orientation as \( (b'_i, b'_j, a_1) \) and \( (b'_i, b'_j, a_2) \). Similarly, we place two points \( a_3 \) and \( a_4 \) sufficiently close to \( p^- \) so that \( a_3 \) is to the left of \( a_4 \), the segment \( a_3 a_4 \) lies on the line \( p^- q \) and has \( p^- \) as its midpoint, the point \( a_4 \) is to the left of \( b'_n \), and all non-collinear triples \( (b'_i, b'_j, p^-) \) have the same orientation as \( (b'_i, b'_j, a_3) \) and \( (b'_i, b'_j, a_4) \). Figure 12 gives an illustration.

![Figure 12: The set C constructed in the proof of Proposition 23 for n odd.](image)

We let \( A, B' \), and \( B'_3 \) be the sets \( \{a_1, a_2, a_3, a_4\}, \{b'_1, \ldots, b'_n\} \), and \( B' \setminus \{b'_3\} \), respectively. Note that the line \( \overline{a_3 a_4} \) intersects the segment \( b'_1 b'_3 \). Since \( \max_{a \in A} x(a) < \min_{b' \in B'} x(b') \), the sets \( A \) and \( B' \) are separated by a vertical line \( \ell \).

Next we slightly perturb \( b'_3 \) to obtain a point \( b_3 \) such that \( b_3 \) lies above \( \ell^- \) and all non-collinear triples \( (b_3, c, d) \) with \( c, d \in A \cup B'_3 \) have the same orientation as \( (b'_3, c, d) \). Note that the point \( b_3 \) lies in the interior of \( \text{conv}(B'_3) \), since \( n \geq 5 \).

To ensure general position, we transform every point \( b'_i = (x, y) \in B'_3 \cap \ell^+ \) to \( b_i = (x, y - \varepsilon x^2) \) and every point \( b'_i = (x, y) \in B'_3 \cap \ell^- \) to \( b_i = (x, y + \varepsilon x^2) \) for some \( \varepsilon > 0 \). The remaining points in \( A \cup \{b_3\} \) remain unchanged. We choose \( \varepsilon \) sufficiently small so that all non-collinear triples of points from \( A \cup B'_3 \cup \{b_3\} \) have the same orientations as their images after the perturbation. Finally, let \( B \) be the set \( \{b_1, \ldots, b_n\} \) and set \( B_3 := B \setminus \{b_3\} \).

Since the points from \( B_3 \) lie on two parabolas, the set \( B \) is in general position. In par-
ticular, points from $B_3$ are in convex position and the point $b_3$ lies inside conv$(B_3)$. Also observe that the line $\ell$ separates $A$ and $B$ and that $a_1$, $a_3$, and $b_1$ are the extremal points of $C := A \cup B$. Since neither of the sets $A$ and $B$ is in convex position, and removal of any of the extremal points $a_1, a_3, b_1$ leaves either $A$ or $B$ in convex position, the set $C = A \cup B$ is $\ell$-critical.

We now show that $C$ contains no $\ell$-divided 5-hole. Suppose for contradiction that there is an $\ell$-divided 5-hole $H$ in $C$. We set $A^+ := \{a_1, a_2\}$, $A^- := \{a_3, a_4\}$, $B^+ := \{b_2, b_4, \ldots, b_{n-1}\}$, and $B^- := \{b_1, b_3, \ldots, b_n\}$. First we assume that $H$ contains points from both $A^+$ and $A^-$. Then $H \cap B \subseteq \{b_{n-1}, b_n\}$, since if there is a point $b_i$ in $H$ with $i < n - 1$, then $b_n$ lies in the interior of conv$(H)$. Note that if $H \cap B = \{b_{n-1}, b_n\}$, then neither $a_4$ nor $a_1$ lies in $H$ and thus $|H| < 5$. Hence $|H \cap B| = 1$, which is again impossible, as $H$ cannot contain all points from $A$. Therefore we either have $H \cap A \subseteq A^+$ or $H \cap A \subseteq A^-$ and, in particular, $1 \leq |H \cap A| \leq 2$.

We now distinguish the following two cases.

1. $|H \cap A| = 2$. If $H \cap A = A^+$, then the hole $H$ can contain only the point $b_n$ from $B^-$. This is because if there is a point $b_i$ in $H \cap B^-$ with $i < n$, then the point $b_{i+1}$ lies in the interior of conv$(H)$. Additionally, $H$ contains at most two points from $B^+$, since otherwise $H$ is not in convex position. Consequently, $b_n$ lies in $H$ and $|H \cap B^+| = 2$, which is impossible, as $H$ would not be in convex position.

If $H \cap A = A^-$, then the hole $H$ contains no point from $B^+$. This is because if there is a point $b_i$ in $H \cap B^+$, then the point $b_{i+1}$ lies in the interior of conv$(H)$. The point $b_i$ cannot lie in $H$ because otherwise $H$ is not in convex position as the line $\overline{a_3a_1}$ separates $b_i$ from $B \setminus \{b_1\}$. Additionally, $H$ contains at most two points from $B^-$, since otherwise $H$ is not in convex position. Thus $H$ contains at most four points of $C$, which is impossible.

2. $|H \cap A| = 1$. Assume first that $H \cap A \subseteq A^+$. Note that for $b_i, b_j \in B^-$ with $i < j \leq n$, the point $b_{i+1}$ lies inside the triangle $\triangle(a_1, b_i, b_j)$ and, if $j < n$, the point $b_{j+1}$ lies inside $\triangle(a_2, b_i, b_j)$. Thus $H$ contains at most one point from $B^-$ or we have $H \cap B^- = \{b_{n-2}, b_n\}$ and $H \cap A = \{a_2\}$. The latter case does not occur, since for every $b_i \in B^+$ with $i < n - 1$ the point $b_{n-1}$ lies in the interior of conv$(\{a_2, b_i, b_{n-2}, b_n\})$. Therefore we consider the case $|H \cap B^-| \leq 1$. However, $|H \cap B^+| \geq 3$ is impossible since $H$ would not be in convex position. Altogether, we obtain $|H| < 5$, which is impossible.

Now we assume that $H \cap A \subseteq A^-$. Note that for $b_i, b_j \in B^+$ with $i < j < n$, the point $b_{i+1}$ lies inside the triangle $\triangle(a_4, b_i, b_j)$ and the point $b_{j+1}$ lies inside $\triangle(a_3, b_i, b_j)$. Thus $H$ contains at most one point from $B^+$. Consequently, $H$ contains at least three points from $B^-$, which is possible only if $H \cap B^- = \{b_1, b_3, b_5\}$. However, then $H$ contains a point $b_i$ from $B^+$ and $b_3$ lies in the interior of conv$(H)$.

Thus, in any case, $H$ is not an $\ell$-divided 5-hole in $C$, a contradiction.

To finish the proof, we consider the case where $n$ is even. Let $\tilde{C} = A \cup \tilde{B}$ be the set constructed above with $|A| = 4$ and $|	ilde{B}| = n + 1$. We set $B := \tilde{B} \setminus \{b_2\}$ and $C := A \cup B$. Note that $C$ is $\ell$-critical.

It remains to show that $C$ contains no $\ell$-divided 5-hole. Suppose for contradiction that there is an $\ell$-divided 5-hole $H$ in $C$. There is no $\ell$-divided 5-hole in $\tilde{C}$ and thus $b_2$ lies in the interior of conv$(H)$. Since $b_1$ is the only point from $C$ to the right of $b_2$, the point $b_1$ lies in $H$. 

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Since $a_1$ is the only point of $C$ to the left of $\overline{b_2b_1}$, all other points of $H$ lie to the right of $\overline{b_2b_1}$. Then, however, the set $(H \setminus \{a_1\}) \cup \{b_2\}$ is a 5-hole in $\tilde{C}$, which gives a contradiction.

6.2 Necessity of the assumptions in Lemmas 14 to 17

We remark that all the assumptions in the statements of Lemmas 14 to 17 are necessary; Figure 13(a) shows that the conditions $|B| = 5$ in Lemma 16 and the convexity of $A$ in Lemma 17 are both necessary. The horizontal reflection of Figure 13(a) also shows the necessity of the assumption $|A| = 5$ in Lemma 14. It follows from the example in Figure 13(b) that the condition $|B| = 4$ cannot be omitted in Lemma 17, since there is an $a$-wedge with three points of $B$. The same point set without the point $a'$ shows that the assumption $|B| \geq 4$ in Lemma 15 is necessary. The example from Figure 13(c) shows that the conditions $|B| = 6$ in Lemma 14, the convex position of $A$ in Lemma 15, and $|A| = 6$ in Lemma 16 are all necessary. The same set without the point $a$ shows that $|A| = 5$ in Lemma 15, and if we remove the points $a$ and $a'$, then the resulting point set shows that we need $5 \leq |A|$ in Lemma 17. We can make statements only about convex $a$-wedges in Lemmas 15 and 16, as there are counterexamples for the corresponding statements without the convexity condition. It suffices to consider so-called double-chains, which are point sets obtained by placing $n$ points on each of the two branches of a hyperbola. Double-chains also show that $A$ cannot be in convex position in Lemma 14, and, that the non-convex $a$-wedge must be empty of points in $B$ in Lemma 17.

**Figure 13:** Examples of point sets that witness tightness of Lemmas 14 to 17. All $k$-holes in these sets with $k \geq 5$ are highlighted in gray.

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A Flow summary

Figure 14: Flow summary. The shaded boxes correspond to computer-assisted results.